

On second order linear divisibility sequences over algebraic number fields

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To the memory of B. Barna, K. Buzási, S. Buzási and M. Erdélyi

1. Introduction

Let R be an integral domain which is finitely generated over $\mathbf{Z}$. (Linear) *divisibility sequences* over R are recurrence sequences $\{u_h\}_{h=0}^{\infty}$, $u_h \in R$, $h = 0, 1, \dots$ (that is sequences in R satisfying linear homogeneous recurrence relations with constant coefficients) with the property that whenever $h|k$, then $u_h|u_k$ in R . This notation was introduced by M. HALL [4] who described all second order divisibility sequences over $\mathbf{Z}$, as well as the third order divisibility sequences over $\mathbf{Z}$ having irreducible characteristic polynomials.

Let $d > 1$ be an integer. The recurrence sequence $\{u_h\}_{h=0}^{\infty}$ is called d -(linear) *divisibility sequence* if $u_h|u_{hd}$ in R for $h = 0, 1, 2, \dots$. For $R = \mathbf{Z}$, SOLOMON [6] characterized all 2-divisibility sequences. BÉZIVIN, PETHŐ and VAN DER POORTEN [1] proved for any R , that if $\{u_h\}_{h=0}^{\infty}$ is d -divisible for an integer $d > 1$ then there is a recurrence sequence $\{\bar{u}_h\}_{h=0}^{\infty}$ of the form

$$\bar{u}_h = h^k \prod_i \left(\frac{\alpha_i^h - \beta_i^h}{\alpha_i - \beta_i} \right)$$

with some integer $k \geq 0$ over R such that $u_h|\bar{u}_h$ in R for $h = 0, 1, 2, \dots$. Thus they confirmed an old conjecture of WARD [7]. For further references concerning divisibility sequences, we refer to [1].

Although the result of Bézivin et al. is very general, it is not straightforward to deduce from it a complete list of d -divisibility sequences over a

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given ring. The aim of this paper is to give a more explicit description of second order d -divisibility recurrence sequences over the ring of integers $\mathbb{Z}_K$ of an algebraic number field K (cf. Theorem 1). In fact, we shall give a criterion for second order non-degenerate recurrence sequences over $\mathbb{Z}_K$ to be d -divisible for some integer $d > 1$. Further, we show (cf. Corollary 2) that a second order non-degenerate recurrence sequence over $\mathbb{Z}_K$ is a divisibility sequence if and only if it is 2-divisible (Corollary 2). Finally, using Theorem 1 we give explicitly all second order recurrence sequences over $\mathbb{Z}$ (cf. Theorem 2) which are d -divisible for some $d > 1$.

2. Results

To state our results we need some notations. Let K be an algebraic number field and denote by $\mathbb{Z}_K$ its ring of integers. Let the sequence $\{u_h\}_{h=0}^\infty$ be defined by the initial terms u_0, u_1 and by the recursion

$$(1) \quad u_{n+2} = Au_{n+1} + Bu_n, \quad n \geq 0,$$

where $u_0, u_1, A, B \in \mathbb{Z}_K$ and $u_0^2 + u_1^2 \neq 0$, $B \neq 0$. Denote by α and β the zeros of the polynomial $x^2 - Ax - B$. Then we have (see e.g. [5])

$$(2) \quad u_n = a\alpha^n - b\beta^n \quad \text{for } n = 0, 1, 2, \dots$$

or

$$(3) \quad u_n = (an + b)\alpha^n \quad \text{for } n = 0, 1, 2, \dots,$$

according as $\alpha \neq \beta$ or $\alpha = \beta$. Further, we have $a = \frac{u_1 - u_0\beta}{\alpha - \beta}$ and $b = \frac{u_1 - u_0\alpha}{\alpha - \beta}$ in (2), and $a = \frac{u_1 - u_0\alpha}{\alpha}$ and $b = u_0$ in (3).

The sequence $\{u_h\}_{h=0}^\infty$ is called *degenerate* if α/β is a root of unity, and *non-degenerate* otherwise.

Theorem 1. Let $\{u_h\}_{h=0}^\infty$ be a second order non-degenerate recurrence sequence over $\mathbb{Z}_K$ with the parameters specified above. (A) If there exists an integer $d > 1$ and an n_0 such that $u_n | u_{nd}$ in $\mathbb{Z}_K$ for all $n \geq n_0$, then $b^{d-1} = a^{d-1}$. (B) Conversely, if $b^{d-1} = a^{d-1}$ for some integer $d > 1$, then $\{u_h\}$ is d -divisible.

In other words, under the assumptions of Theorem 1 $\{u_h\}$ is d -divisible if and only if $b^{d-1} = a^{d-1}$.

If the assumptions in (A) of Theorem 1 hold for $d = 2$ then we get that $b = a$. Further, in this case $u_0 = 0$, $b = a = \frac{u_1}{\alpha - \beta}$ and hence $\{u_h\}$ is divisible. Conversely, if $u_0 = 0$ then $b = a$. Thus we have the following

Corollary 1. Let $\{u_h\}_{h=0}^\infty$ be as in Theorem 1. (A) Assume that there exists an n_0 such that $u_n | u_{2n}$ in $\mathbf{Z}_K$ for all $n \geq n_0$. Then $u_0 = 0$ and

$$u_n = u_1 \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{for } n = 0, 1, 2, \dots$$

(B) Conversely, if $u_0 = 0$ then $\{u_h\}$ is divisible.

This implies the following

Corollary 2. Let $\{u_h\}_{h=0}^\infty$ be a second order non-degenerate recurrence sequence over $\mathbf{Z}_K$. $\{u_h\}$ is a divisibility sequence if and only if it is 2-divisible.

The degree of $K(\alpha)$ over $\mathbf{Q}$ is at most $2k$, where k denotes the degree of K over $\mathbf{Q}$. Hence, in Theorem 1, there exist only finitely many possibilities for a/b (if $b \neq 0$) and b/a (if $a \neq 0$) which are easily computable if K is given. We shall carry out this explicitly only for $K = \mathbf{Q}$. Using Theorem 1, we shall list in Theorem 2 below all second order (degenerate and non-degenerate) recurrence sequences over $\mathbf{Z}$ which are d -divisible for some $d > 1$.

Theorem 2. Let $\{u_h\}_{h=0}^\infty$ be a second order recurrence sequence over $\mathbf{Z}$ with the parameters $u_0, u_1, A, B \in \mathbf{Z}$, $u_0^2 + u_1^2 \neq 0$, $B \neq 0$ specified above, and let $d > 1$ be an integer. The sequence $\{u_h\}$ is d -divisible if and only if there exist $e, f \in \mathbf{Z}$ such that at least one of the following cases holds:

- (i) $u_0 = 0$, d arbitrary;
- (ii) $A = 0$, d odd;
- (iii) $A = 0$, $u_1 | u_0 B^{d/2}$, d even;
- (iv) $u_0 A = 2u_1$, $A^2 + 4B = 0$, d arbitrary;
- (v) $u_0 A = 2u_1$, d odd;
- (vi) $A = 2e$, $B = -2e^2$, $e \neq 0$, $d \equiv 1 \pmod{4}$;
- (vii) $A = 2e$, $B = -2e^2$, $e \neq 0$, $u_r | B^{2t} u_{r_0}$ for all integers r, t, r_0 with $1 \leq r \leq 3$, $dr = 4t + r_0$, $t \geq 0$, $0 \leq r_0 \leq 3$ and $d \not\equiv 1 \pmod{4}$;
- (viii) $A = 2e$, $B = -(e^2 + f^2)$, $f \neq 0$, $e \neq \pm f$, $u_1 = u_0(e \pm f)$ and $d \equiv 1 \pmod{4}$;
- (ix) $A = -f$, $B = -f^2$, $f \neq 0$, $d \equiv 1 \pmod{3}$;
- (x) $A = -f$, $B = -f^2$, $f \neq 0$, $u_r | f^{3t} u_{r_0}$ for all integers r, t, r_0 with $1 \leq r \leq 2$, $dr = 3t + r_0$, $t \geq 0$, $0 \leq r_0 \leq 2$ and $d \not\equiv 1 \pmod{3}$;
- (xi) $A = 2e - f$, $B = -(e^2 - ef + f^2)$, $e \neq 0$, $f \neq 0$, $e \neq \pm f$, $u_0 \neq 0$, $u_1/u_0 \in \{e - f, e\}$, $d \equiv 1 \pmod{3}$;
- (xii) $A = 3f$, $B = -3f^2$, $f \neq 0$, $d \equiv 1 \pmod{6}$;
- (xiii) $A = 3f$, $B = -3f^2$, $f \neq 0$, $u_r | B^{3t} u_{r_0}$ for all integers r, t, r_0 with

- (xiv) $1 \leq r \leq 5$, $dr = 6t + r_0$, $t \geq 0$, $0 \leq r_0 \leq 5$ and $d \not\equiv 1 \pmod{6}$;
 $A = 2e - f$, $B = -(e^2 - ef + f^2)$, $e \neq \pm f$, $2f$, $u_0 \neq 0$,
 $u_1/u_0 \in \{e + f, e - 2f\}$ and $d \equiv 1 \pmod{6}$.

It follows from Theorem 2 (see also its proof) that in cases (ii) – (iv), (vi), (vii), (ix), (x), (xii) and (xiii), the sequence $\{u_h\}$ is degenerate. In case (i), it is easy to give an example both for the degenerate and for the non-degenerate case. In the other cases $\{u_h\}$ is non-degenerate. More precisely, it is easy to deduce from Theorems 1, 2 and Corollary 2 that there exist only the following five types of second order non-degenerate d -divisibility sequences $\{u_h\}$ over $\mathbf{Z}$;

- $\{u_h\}$ is 2-divisible; then it is divisible (case (i));
- $\{u_h\}$ is 2-non-divisible but 3-divisible; then it is $(2k + 1)$ -divisible for every $k \in \mathbf{N}$ (case (v));
- $\{u_h\}$ is 2-non-divisible but 4-divisible; then it is $(3k + 1)$ -divisible for every $k \in \mathbf{N}$ (case (xi));
- $\{u_h\}$ is 3-non-divisible but 5-divisible; then it is $(4k + 1)$ -divisible for every $k \in \mathbf{N}$ (case (viii));
- $\{u_h\}$ is 3 and 4-non-divisible but 7-divisible; then it is $(6k + 1)$ -divisible for every $k \in \mathbf{N}$ (case (xiv)).

3. Proofs

PROOF of Theorem 1. First we prove assertion (A). Let $\{u_h\}$ be a second order non-degenerate recurrence sequence over $\mathbf{Z}_K$, satisfying the assumptions made in (A) of Theorem 1. Using the notations of Section 2 we have $\alpha \neq \beta$, hence u_n satisfies (2).

An easy computation shows that

$$(4) \quad (a\alpha^n - b\beta^n) \frac{\alpha^{dn} - \beta^{dn}}{\alpha^n - \beta^n} = a\alpha^{dn} - b\beta^{dn} + (a-b)(\alpha\beta)^n \frac{\alpha^{(d-1)n} - \beta^{(d-1)n}}{\alpha^n - \beta^n}$$

for all integers $n, d \geq 1$. Put $L = K(\alpha)$, and denote by $\mathbf{Z}_L$ the ring of integers of L . It is clear that

$$\frac{\alpha^{jn} - \beta^{jn}}{\alpha^n - \beta^n} \in \mathbf{Z}_K \quad \text{for every integer } j \geq 1.$$

Hence, if $u_n | u_{dn}$ then, by (4),

$$(a-b)(\alpha\beta)^n \frac{\alpha^{(d-1)n} - \beta^{(d-1)n}}{\alpha^n - \beta^n} \in \mathbf{Z}_K$$

and

$$(5) \quad u_n \mid (a-b)(\alpha\beta)^n(\alpha^{(d-1)n} - \beta^{(d-1)n}) \quad \text{in } \mathbf{Z}_L.$$

Further, we note that $a-b = u_0$, whence $a-b \in \mathbf{Z}_K$.

For a non-zero ideal $\mathcal{A}$ of $\mathbf{Z}_L$, let $P(\mathcal{A})$ denote the maximum of the rational primes lying below the prime ideal divisors of $\mathcal{A}$. Further, we put $P(0) = P(1) = 1$. By a result of MAHLER [3], $P(u_n)$ is not bounded as $n \rightarrow \infty$. Hence there exists a number n_1 such that $P(u_n) > \max\{P(a-b), P(\alpha\beta)\}$ for infinitely many $n \geq n_1$. Let $n \geq \max\{n_0, n_1\}$ with this property, and let $\wp_n$ be a prime ideal divisor of u_n with $P(\wp_n) = P(u_n)$. Then it follows from (5) that

$$(6) \quad \wp_n \mid \alpha^{(d-1)n} - \beta^{(d-1)n} \quad \text{in } \mathbf{Z}_L.$$

Put

$$A_i = (\alpha - \beta)^{i-1}(b^{i-1}\alpha^{(d-i)n} - a^{i-1}\beta^{(d-i)n}) \quad \text{for } i = 1, \dots, d.$$

Then $A_i \in \mathbf{Z}_L$ for each i . Further, $b(\alpha - \beta) \in \mathbf{Z}_L$. It is easy to see that

$$(7) \quad (\alpha - \beta)^{i-1}\beta^{(d-i)n}a^{i-2}(a\alpha^n - b\beta^n) - b(\alpha - \beta)A_{i-1} = -\alpha^n A_i$$

for each integer i with $2 \leq i \leq d$. We prove now that

$$(8) \quad \wp_n \mid A_i \quad \text{in } \mathbf{Z}_L$$

for each integer i with $1 \leq i \leq d$. By (6), (8) is true for $i = 1$. Assume that it is true for $i - 1 \geq 0$. Then by the definition of $\wp_n$ and by the induction hypothesis, $\wp_n$ divides the element on the left-hand side of (7). Since $\wp_n \nmid \alpha^n$ in $\mathbf{Z}_L$, it must divide A_i in $\mathbf{Z}_L$ on the right-hand side of (7), and (8) is proved.

Setting $i = d$ in (8) we get

$$\wp_n \mid (\alpha - \beta)^{d-1}(b^{d-1} - a^{d-1}) \quad \text{in } \mathbf{Z}_L.$$

But $P(\wp_n)$ can be arbitrarily large, hence $b^{d-1} = a^{d-1}$ which proves the assertion in (A).

Conversely, suppose now that $b^{d-1} = a^{d-1}$ for some integer $d > 1$. Then $b = \zeta_{d-1}a$ with some $(d-1)$ -th root of unity ζ_{d-1} from L . Then, by (2), we have

$$(9) \quad \begin{cases} u_{dn} = a(\alpha^{dn} - \zeta_{d-1}\beta^{dn}) = a((\alpha^n)^d - (\zeta_{d-1}\beta^n)^d) = u_n v_n & \text{for all } n \geq 0, \\ \text{where} \\ v_n = \begin{cases} (\alpha^n)^{d-1} + (\alpha^n)^{d-2}(\zeta_{d-1}\beta^n) + \dots + (\zeta_{d-1}\beta^n)^{d-1} & \text{for } n \geq 1, \\ 1 & \text{for } n = 0. \end{cases} \end{cases}$$

In (9), v_n is an algebraic integer. On the other hand, if $u_n \neq 0$ then, by (9), v_n belongs to K , hence $v_n \in \mathbb{Z}_K$. Thus $\{u_h\}$ is d -divisible which proves the assertion of (B).

PROOF of Theorem 2. Let now $\{u_h\}_{h=0}^\infty$ be a second order recurrence sequence over $\mathbb{Z}$ with initial terms $u_0, u_1 \in \mathbb{Z}$, $u_0^2 + u_1^2 \neq 0$, which satisfies (1) with some $A, B \in \mathbb{Z}$, $B \neq 0$, and suppose that it is d -divisible for some integer $d > 1$. Define α, β, a, b as in Section 2.

Assume first that $\{u_h\}$ is degenerate. Then $\alpha = \zeta\beta$, where ζ is a root of unity belonging to $K = \mathbb{Q}(\alpha)$. The degree of K over $\mathbb{Q}$ is at most two, hence $\zeta \in \mathcal{E} = \{\pm 1, \pm i, \pm \rho, \pm \rho^2\}$ where $\rho = \frac{-1+\sqrt{-3}}{2}$ (see e.g. [2]). We shall distinguish several cases.

Case 1. If $\alpha = \beta$, then $\alpha \in \mathbb{Z}$ and u_n may be written in the form (3). Further, $A^2 + 4B = 0$ and $\alpha = A/2$. If $a = 0$ then $u_0 A = 2u_1$ which corresponds to case (iv). This sequence is indeed d -divisible for every $d > 1$. If $b = 0$ then $u_0 = 0$ which corresponds to case (i). Then the sequence is d -divisible. Finally, suppose that a and b are different from zero. Then $\alpha(2nd + b)$ and $\alpha(an + b)$ are rational integers and $u_n | u_{dn}$ implies that $\alpha(an + b) | \alpha(2nd + b)$. From this it follows that $\alpha(an + b) | \alpha b(d-1)$ which is impossible if n is large enough.

Case 2. Let now $\alpha = -\beta$. Then $A = 0$ and $B = \beta^2$. We have $u_{2n} = u_0 \beta^{2n}$ and $u_{2n+1} = u_1 \beta^{2n}$ for $n = 0, 1, \dots$. If d is odd then $dn \equiv n \pmod{2}$, hence, indeed, $u_n | u_{dn}$ and (ii) follows. While if d is even and n is odd then $u_1 \beta^{n-1} | u_0 \beta^{dn}$ must hold. Then $\{u_h\}$ is indeed d -divisible and $\{u_h\}$ is described in case (iii).

Case 3. Let $\alpha = \pm i\beta$. Then $\alpha + \beta = \beta(1 \pm i) = A$ implies that $\beta = \frac{A}{2}(1 \mp i)$. Since β is an algebraic integer, we have $A = 2e$ and $B = -\alpha\beta = -2e^2$. Let n be an arbitrary non-negative integer, and put $n = 4v + r$ with non-negative integers v, r such that $0 \leq r < 4$. Then, by (2),

$$(10) \quad \begin{aligned} u_n &= a\alpha^n - b\beta^n = \beta^n(a(\pm i)^n - b) = \\ &= e^n(1 \mp i)^n(a(\pm i)^n - b) = (-1)^v B^{2v} u_r. \end{aligned}$$

If $d \equiv 1 \pmod{4}$ then $dn \equiv n \equiv r \pmod{4}$ and, by (10), $u_n | u_{dn}$, i.e. $\{u_h\}$ is indeed d -divisible, and this is case (vi). If $d \not\equiv 1 \pmod{4}$ then we put $dr = 4t + r_0$ with non-negative integers t, r_0 such that $0 \leq r_0 < 4$. Then $dn = 4vd + 4t + r_0$. By (10), in this case $\{u_h\}$ is d -divisible if and only if

$$(11) \quad u_r | B^{2(v(d-1)+t)} u_{r_0} \quad \text{for all } v \geq 0 \text{ and each } r, t, r_0 \text{ such that } 0 \leq r, r_0 < 4 \text{ and } dr = 4t + r_0.$$

But for fixed r, r_0 and t , (11) holds for all $v \geq 0$ if and only if it holds for $v = 0$. Finally, (11) trivially holds for $r = 0$, hence we get case (vii).

Case 4. Next let $\alpha/\beta = \rho^j$ where $j = 1$ or 2 . Then α and β belong to the ring of integers of the Eulerian number field $\mathbf{Q}(\rho)$. In this field, $\{1, \rho\}$ is an integral basis, hence we can write $\alpha = e + (-1)^j f \rho$ and $\beta = e + (-1)^j f \rho^2$ with some $e, f \in \mathbf{Z}$. Then, by $\alpha = \beta \rho^j$, we get $A = -f$, $B = -f^2$ and $\beta = \rho^j f$, $j = 1, 2$. Put $n = 3v + r$ with integers v, r such that $v \geq 0$, $0 \leq r < 3$. Then, by (2) and $(\rho^j)^3 = 1$, we have

$$u_n = \beta^n (a \rho^{jn} - b) = f^{3v} \cdot f^r \cdot \rho^{jr} (a \rho^{jr} - b) = f^{3v} u_r.$$

From now on we can proceed in a similar way as in the cases corresponding to $\alpha/\beta = \pm i$, and we get (ix) and (x) in the theorem.

Case 5. Let now $\alpha/\beta = -\rho^j$ where $j = 1$ or 2 . Then we get in the same way as in the preceding case that $A = 3f$, $B = -3f^2$ and $\beta = \sqrt{-3} \cdot f(-\rho)^j$ with some rational integer $f \neq 0$. Let $n = 6v + r$ with rational integers v, r such that $v \geq 0$, $0 \leq r < 6$. Then $(-\rho^j)^6 = 1$ and (2) imply again that

$$u_n = \beta^n (a(-\rho^j)^n - b) = (\sqrt{-3}f)^{6v} u_r = B^{3v} u_r.$$

Now we can proceed as in the case above $\alpha/\beta = \pm i$, and we get cases (xii), (xiii) in our theorem.

In the sequel we suppose that $\{u_h\}_{h=0}^\infty$ is non-degenerate. Since $\{u_h\}$ is a second order recurrence sequence, we have $ab \neq 0$. By Theorem 1, $\{u_h\}$ is d -divisible if and only if $b = \zeta a$ with some $(d-1)$ -th root of unity ζ . Then $\zeta \in \mathbf{Q}(\alpha)$, hence $\zeta \in \mathcal{E}$. Further, $b = \zeta a$ is equivalent to

$$(12) \quad (u_1 - u_0 \beta) \zeta = u_1 - u_0 \alpha.$$

It suffices to determine all second order non-degenerate recurrence sequences $\{u_h\}$ in $\mathbf{Z}$ having the property (12). We shall distinguish again several cases.

Case 6. Let first $\zeta = 1$. Then (12) implies $u_0 = 0$ and $\{u_h\}$ is d -divisible for every $d > 1$. This corresponds to (i) in the theorem.

Case 7. $\zeta = -1$. This appears only if d is odd. Then we get from (12) that

$$2u_1 = u_0(\alpha + \beta) = u_0 A$$

because α and β are the zeros of $x^2 - Ax - B$. This is the case (v) in the theorem.

Case 8. $\zeta = \pm i$. This is possible only if $d \equiv 1 \pmod{4}$.

Then $K = \mathbf{Q}(\alpha) = \mathbf{Q}(i)$ is the Gaussian number field. Hence $\alpha = e + if$ and $\beta = e - if$ with suitable $e, f \in \mathbf{Z} \setminus \{0\}$ for which $e \neq \pm f$.

For $\zeta = i$ and $\zeta = -i$, (12) implies $u_1 = u_0(e - f)$ and $u_1 = u_0(e + f)$, respectively. Furthermore, we have in both cases $A = 2e$ and $B = -(e^2 + f^2)$ which corresponds to case (viii) of the theorem.

Case 9. $\zeta = \rho^j$ where $j = 1$ or 2 . Then $d \equiv 1 \pmod{3}$ and $K = \mathbf{Q}(\alpha) = \mathbf{Q}(\rho)$ is the Eulerian number field. Thus $\alpha = e + f\rho$ and $\beta = e + f\rho^2$ with suitable integers $e, f \in \mathbf{Z} \setminus \{0\}$ for which $e \neq \pm f, 2f$. Hence $A = 2e - f$ and $B = -(e^2 - ef + f^2)$. Using again (12), we get

$$u_1 = \begin{cases} u_0(e - f) & \text{if } \zeta = \rho \\ u_0e & \text{if } \zeta = \rho^2, \end{cases}$$

which corresponds to (xi) in the theorem.

Case 10. $\zeta = -\rho^j$ where $j = 1$ or 2 . Then $d \equiv 1 \pmod{6}$ and $K = \mathbf{Q}(\alpha) = \mathbf{Q}(\rho)$. Taking again $\alpha = e + f\rho$ and $\beta = e + f\rho^2$ with some $e, f \in \mathbf{Z} \setminus \{0\}$, we get for e, f the restrictions $e \neq \pm f, 2f$. A and B have the same form as in case 9. Finally, using (12) we get

$$u_1 = \begin{cases} u_0(e + f) & \text{if } \zeta = -\rho \\ u_0(e - 2f) & \text{if } \zeta = -\rho^2, \end{cases}$$

which corresponds to case (xiv) in the theorem. This completes the proof of Theorem 2.

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