

On the geometry of generalized metric spaces II. Spaces of isotropic curvature

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§0. Introduction

Let M be an n -dimensional differentiable manifold and $T(M)$ its tangent bundle. We consider the bundle M_T which does not contain zero vector of $T(M)$, that is, $M_T := T(M) - \{0\}$. A generalized metric space M_n (or a generalized Finsler space) is a pair $(M_T, g_{ij}(x, y))$, where $g_{ij}(x, y)$ is a symmetric tensor satisfying the following conditions:

(a) $g_{ij}(x, y)$ is positively homogeneous of degree zero in y^i , (b) g_{ij} is positive definite, (c) $g^*_{ij} := \frac{1}{2} \partial^2 F^2 / \partial y^i \partial y^j$ is non-degenerate, where $F^2 := g_{ij} y^i y^j$.

In the previous paper [1] we introduced three types of connections in M_n (see §1): $CT(N) = (F_j^i{}_k, C_j^i{}_k)$, $R\Gamma(N) = (F_j^i{}_k, 0)$ and $B\Gamma(G) = (G_j^i{}_k, 0)$ and the curvature tensors are $(R_h^i{}_{jk}, P_h^i{}_{jk}, S_h^i{}_{jk})$ for $CT(N)$, $(K_h^i{}_{jk}, F_h^i{}_{jk}, 0)$ for $R\Gamma(N)$ and $(H_h^i{}_{jk}, G_h^i{}_{jk}, 0)$ for $B\Gamma(G)$ respectively.

Let κ_{hijk} be one of the above seven curvature tensors of M_n . For the plane $\pi(X, Y)$ spanned by two independent vectors X^i and Y^i in M_n , the scalar

$$(0.1) \quad \kappa(x, y, \pi(X, Y)) := \kappa_{hijk} X^h Y^i X^j Y^k / (g_{hj} g_{ik} - g_{hk} g_{ij}) X^h Y^i X^j Y^k$$

is called the *sectional curvature* for $\pi(X, Y)$ with respect to κ_{hijk} . If the sectional curvature κ is independent of any X^i and Y^i , then the space $M_n (n > 2)$ is called to be *isotropic with respect to κ_{hijk}* , or of *κ -isotropic curvature*. We assume that the scalar κ does not vanish.

The purpose of the present paper is to investigate the properties on generalized metric spaces of R -, K - and H -isotropic curvature.

We shall show that a generalized metric space of K -isotropic curvature is a Riemannian space of constant curvature (§3) and a generalized metric space of H -isotropic curvature is a Finsler space of constant curvature (§4).

It seems that R -isotropic curvature on a generalized metric space has not yet been sufficiently investigated.

The terminology and notations are the same as in the paper [1] unless otherwise stated.

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§1. Preliminaries

In this section we shall quote some quantities and relations from the paper [1] for later use.

Let M_n be a generalized metric space with tensor $g_{ij}(x, y)$ mentioned in §0. As usual we raise or lower indices by means of g_{ij} .

Three types of connection forms defined on M_n are given as follows:

$$\begin{aligned}\omega_j^i &= F_j^i{}_k dx^k + C_j^i{}_k \delta y^k, \quad \delta y^k := dy^k + N^k{}_m dx^m \quad \text{for } C\Gamma(N), \\ \omega_j^i &= F_j^i{}_k dx^k \quad \text{for } R\Gamma(N) \quad \text{and} \quad \omega_j^i = G_j^i{}_k dx^k \quad \text{for } B\Gamma(G),\end{aligned}$$

where

$$\begin{aligned}F_j^i{}_k &:= \frac{1}{2} g^{ih} (d_j g_{hk} + d_k g_{hj} - d_h g_{jk}), \quad d_k := \partial_k - N^h{}_k \dot{\partial}_h, \quad \partial_k := \partial / \partial x^k, \\ \dot{\partial}_h &:= \partial / \partial y^h, \quad C_j^i{}_k := \frac{1}{2} g^{ih} (g_{hk(j)} + g_{hj(k)} - g_{jk(h)}), \quad g_{jk(h)} := \dot{\partial}_h g_{jk}, \\ G_j^i{}_k &:= \dot{\partial}_k G^i{}_j, \quad G^i{}_j := \dot{\partial}_j G^i, \quad G^i := \frac{1}{4} g^{*ih} (y^j \partial_j \dot{\partial}_h F^2 - \partial_h F^2), \\ g^{*ih} g^*_{hj} &= \delta_j^i,\end{aligned}$$

and we have $N^i{}_k = F_j^i{}_k y^j$.

The covariant derivatives of a vector $v^i(x, y)$ are defined as follows:

$$\begin{aligned}(1.1) \quad (a) \quad v^i{}_{/k} &:= d_k v^i + F_j^i{}_k v^j \quad \text{for } C\Gamma(N) \text{ and } R\Gamma(N), \\ (b) \quad v^i{}_{/(k)} &:= v^i{}_{(k)} + C_h^i{}_k v^h \quad \text{for } C\Gamma(N), \\ (c) \quad v^i{}_{//k} &:= \bar{d}_k v^i + G_h^i{}_k v^h, \quad \bar{d}_k := \partial_k - G^h{}_k \dot{\partial}_h \quad \text{for } B\Gamma(G).\end{aligned}$$

Here the following relations are satisfied:

$$\begin{aligned}(1.2) \quad (a) \quad C_0^i{}_k &= C_j^i{}_0 = 0, \quad (b) \quad C_{jk} := C_j^0{}_k = g_{ij(k)} y^i, \\ (c) \quad C_{0k} &= C_{j0} = 0, \quad (d) \quad g_{ij(k)} = C_{ijk} + C_{jik},\end{aligned}$$

where the index 0 denotes transvection by y ;

$$(1.3) \quad v^i_{//k} = v^i_{/k} + D_h^i{}_k v^h - P^h{}_k v^i_{(h)},$$

where $D_h^i{}_k := G_h^i{}_k - F_h^i{}_k$ and $P^h{}_k := G^h{}_k - N^h{}_k = D_0^h{}_k = D_k^h{}_0$;

$$(1.4) \quad (a) P_j^0{}_k = P_j^i{}_0 = 0, \quad (b) P_0^i{}_k = 2P^i{}_k, \quad (c) P^i{}_0 = P^0{}_k = 0,$$

where $P_j^i{}_k := N^i_{j(k)} - F_j^i{}_k$ and

$$(1.5) \quad (a) D_j^0{}_k = -(g_{ik} + C_{ik})P^i{}_j, \quad (b) C_{jk/0} = -2D_j^0{}_k.$$

The connection $CT(N)$ is a metrical connection, that is, $g_{ij/k} = 0$, $g_{ij/(k)} = 0$.

It is known if $C_{ij} = 0$ or $C_{ijk} = 0$, then the generalized metric space reduces to a Finsler space or a Riemannian space respectively.

The so-called Ricci formulae for a vector $v^i(x, y)$ are given as follows:

$$(1.6) \quad \begin{aligned} (a) \quad & v^i_{/j/k} - j|k = R_h^i{}_{jk} v^h - R^h{}_{jk} v^i_{/(h)} = K_h^i{}_{jk} v^h - R^h{}_{jk} v^i_{(h)}, \\ (b) \quad & v^i_{//j/k} - j|k = H_h^i{}_{jk} v^h - H^h{}_{jk} v^i_{(h)}, \end{aligned}$$

where $-j|k$ means the interchange of indices j, k in the foregoing term and subtraction, for instance $A_{jm}B_i^m{}_k - j|k = A_{jm}B_i^m{}_k - A_{km}B_i^m{}_j$. Here the curvature tensors $R_h^i{}_{jk}$, $K_h^i{}_{jk}$, $H_h^i{}_{jk}$ are defined as follows:

$$(1.7) \quad \begin{aligned} (a) \quad & R_h^i{}_{jk} := K_h^i{}_{jk} + C_h^i{}_m R^m{}_{jk}, \\ (b) \quad & K_h^i{}_{jk} := d_k F_h^i{}_j + F_h^m{}_j F_m^i{}_k - j|k, \\ (c) \quad & H_h^i{}_{jk} := \bar{d}_k G_h^i{}_j + G_h^m{}_j G_m^i{}_k - j|k, \end{aligned}$$

and the torsion tensors $R^i{}_{jk}$ and $H^i{}_{jk}$ are defined by

$$(1.8) \quad (a) R^i{}_{jk} := d_k N^i{}_j - j|k, \quad (b) H^i{}_{jk} := \bar{d}_k G^i{}_j - j|k.$$

In this case, the following relations are valid:

$$\begin{aligned} (a) \quad & R^i{}_{jk} = R_0^i{}_{jk} = K_0^i{}_{jk}, \quad H^i{}_{jk} = H_0^i{}_{jk}, \\ (b) \quad & H_h^i{}_{jk} = K_h^i{}_{jk} + E_h^i{}_{jk}, \quad E_h^i{}_{jk} := D_h^i{}_{j/k} + D_h^m{}_j D_m^i{}_k - \\ & - P^m{}_k G_h^i{}_{jm} - j|k, \quad G_h^i{}_{jk} := \dot{\partial}_k G_h^i{}_j, \end{aligned}$$

$$\begin{aligned}
(1.9) \quad & (c) \ H^i_{jk} = R^i_{jk} + E^i_{jk}, \\
& \quad E^i_{jk} := E^i_{0jk} = P^i_{j/k} + P^m_j D^i_{mk} - j|k, \\
& (d) \ H^i_k = R^i_k + E^i_k, \\
& \quad H^i_k := H^i_{0k}, R^i_k := R^i_{0k}, E^i_k := E^i_{0k}, \\
& (e) \ E^0_{jk} = 0, \ E^i_k = -P^i_{k/0} - P^i_m P^m_k, \\
& (f) \ H^i_{j(kh)} = H^i_{hjk}, \ H^i_{k(j)} - H^i_{j(k)} = 3H^i_{jk}.
\end{aligned}$$

From the Bianchi identities, we shall list the following:

$$\begin{aligned}
(1.10) \quad & (a) \ R_h^i{}_{jk} + h|j|k = C_h^i{}_m R^m_{jk} + h|j|k, \quad (b) \ K_h^i{}_{jk} + h|j|k = 0, \\
& (c) \ H_h^i{}_{jk} + h|j|k = 0, \text{ and consequently } (d) \ E_h^i{}_{jk} + h|j|k = 0,
\end{aligned}$$

where $+h|j|k$ means the cyclic permutations of indices h, j, k in the foregoing term and summation, for instance $A_{hm}B_j^m{}_k + h|j|k = A_{hm}B_j^m{}_k + A_{jm}B_k^m{}_h + A_{km}B_h^m{}_j$.

Moreover we have the useful expressions

$$(1.11) \quad R_{hijk} - R_{jkh i} = B_{hijk},$$

$$(1.12) \quad K_{hijk} - K_{jkh i} = \frac{1}{2}(T_{hijk} - g_{hi(m)}R^m_{jk} + g_{jk(m)}R^m_{hi}),$$

$$\begin{aligned}
(1.13) \quad H_{hijk} - H_{jkh i} &= \frac{1}{2}(T_{hijk} - g_{hi(m)}R^m_{jk} + g_{jk(m)}R^m_{hi}) + \\
&\quad + E_{hijk} - E_{jkh i},
\end{aligned}$$

where

$$\begin{aligned}
2B_{hijk} &:= T_{hijk} + (C_{him} - C_{ihm})R^m_{jk} - (C_{jkm} - C_{kjm})R^m_{hi}, \\
T_{hijk} &:= g_{hj(m)}R^m_{ik} + g_{ik(m)}R^m_{hj} - j|k,
\end{aligned}$$

and they satisfy the following relations:

$$\begin{aligned}
(1.14) \quad 2B_{hijk} + h|j|k &= (3C_{him} + C_{ihm})R^m_{jk} - \\
&\quad - (C_{jkm} - C_{kjm})R^m_{hi} + h|j|k,
\end{aligned}$$

$$(1.15) \quad T_{hijk} + h|j|k = 2g_{hi(m)}R^m_{jk} + h|j|k.$$

Lastly, we shall prove the following result:

Lemma 1.1. *If a tensor A_{hijk} of degree 4 satisfies*

$$(1.16) \quad A_{hijk} + A_{jihk} + A_{hkji} + A_{jkh i} = 0,$$

$$(1.17) \quad A_{hijk} = -A_{hikj},$$

$$(1.18) \quad A_{hijk} - A_{jkh i} = U_{hijk},$$

$$(1.19) \quad A_{hijk} + h[j|k = V_{hijk},$$

where U_{hijk} and V_{hijk} are certain tensors, then A_{hijk} is expressible as

$$(1.20) \quad 6A_{hijk} = U_{hijk} + U_{jihk} - U_{kijh} - U_{hikj} + 2V_{hijk}.$$

PROOF. Interchanging indices h and j in (1.18), we get

$$(1.18)' \quad A_{jihk} - A_{hkji} = U_{jihk}.$$

If we take the sum of the three equations (1.16), (1.18) and (1.18)' and use (1.17), we obtain

$$(1.21) \quad 2(A_{hijk} - A_{jihk}) = U_{hijk} + U_{jihk}.$$

The cyclic change $h \rightarrow k \rightarrow j \rightarrow h$ of indices in (1.21) gives

$$(1.21)' \quad 2(A_{kijh} - A_{hikj}) = U_{kijh} + U_{hikj}.$$

Subtracting (1.21)' from (1.21), we have

$$(1.22) \quad 2(2A_{hijk} - A_{jihk} - A_{kijh}) = U_{hijk} + U_{jihk} - U_{kijh} - U_{hikj}.$$

Making use of (1.19) on the left hand side of (1.22), we can see (1.20).
Q.E.D.

§2. A generalized metric space of R -isotropic curvature

First we consider a generalised metric space of R -isotropic curvature. In this case, from the relation corresponding to (0.1) we have

$$(2.1) \quad [R_{hijk} - R(x, y)(g_{hj}g_{ik} - g_{hk}g_{ij})]X^h Y^i X^j Y^k = 0.$$

When we put

$$r_{hijk} := R_{hijk} - R(x, y)(g_{hj}g_{ik} - g_{hk}g_{ij}),$$

provided equation (2.1) holds for any X^i and Y^i , then the following equation must hold:

$$(2.2) \quad r_{hijk} + r_{jihk} + r_{hkji} + r_{jkh i} = 0.$$

On the other hand, the tensor r_{hijk} obeys the following relations using the properties of R_{hijk} :

$$(2.3) \quad r_{hijk} = -r_{hikj},$$

$$(2.4) \quad r_{hijk} - r_{jkh i} = R_{hijk} - R_{jkh i} = B_{hijk},$$

$$(2.5) \quad r_{hijk} + h|j|k = R_{hijk} + h|j|k = C_{him}R^m_{jk} + h|j|k.$$

Consequently, r_{hijk} satisfies the conditions for A_{hijk} in Lemma 1.1, where $U_{hijk} = B_{hijk}$ and $V_{hijk} = C_{him}R^m_{jk} + h|j|k$. Therefore r_{hijk} has the form

$$\begin{aligned} 6r_{hijk} &= B_{hijk} + B_{jihk} - B_{kihj} - B_{hikj} + 2(C_{him}R^m_{jk} + h|j|k) \\ &= 3B_{hijk} - (B_{hijk} - 2C_{him}R^m_{jk} + h|j|k). \end{aligned}$$

Making use of the definition of B_{hijk} and (1.14) in the above equation, we obtain

$$(2.6) \quad \begin{aligned} 6r_{hijk} &= [(C_{ikm} + 2C_{kim})R^m_{hj} + (C_{hjm} + 2C_{jhm})R^m_{ik} - j|k| + \\ &\quad + 2(C_{him} - C_{ihm})R^m_{jk} - (C_{jkm} - C_{kjm})R^m_{hi}. \end{aligned}$$

Consequently, we have the following

Theorem 2.1. *A generalized metric space of R -isotropic curvature is characterized by (2.6).*

Making use of (2.6), we shall prove the following two propositions.

Proposition 2.2. *In a generalized metric space of R -isotropic curvature, if the relation*

$$(2.7) \quad R^i_{jk} = R(y_j h^i_k - y_k h^i_j) \quad (h^i_k := \delta^i_k - y^i y_k / F^2)$$

is satisfied, then the following equation holds:

$$(2.8) \quad R_{hijk} = R(g_{hj}g_{ik} - g_{hk}g_{ij}).$$

PROOF. If we substitute (2.7) into (2.6), then direct calculations show $r_{hijk} = 0$. Hence (2.8) is obtained. Q.E.D.

Proposition 2.3. *In a generalized metric space of R -isotropic curvature, if the symmetric tensor C_{ij} is proportional to h_{ij} , namely*

$$(2.9) \quad C_{ij} = \lambda h_{ij} \quad (\lambda \neq -1),$$

then the following equation holds:

$$(2.10) \quad R^i_{jk} = R(y_j \delta_k^i - y_k \delta_j^i).$$

PROOF. Transvecting (1.7) (a) by y_i and using (1.2) (b), we get $K_h^0{}_{jk} = -(g_{hm} + C_{hm})R^m_{jk}$, transvection of which by y^k gives

$$(2.11) \quad K_{hj} = (g_{hm} + C_{hm})R^m_j,$$

where $K_{hj} := K_h^0{}_{j0}$. Similarly, if we transvect (1.10) (b) by $y_i y^k$, then we obtain

$$(2.12) \quad K_{hj} = K_{jh}.$$

On the other hand, transvecting (2.6) by y^h and using (1.2) (a), (b), we get

$$(2.13) \quad \begin{aligned} 6r_{0ijk} = & (C_{ikm} + 2C_{kim})R^m_j - (C_{ijm} + 2C_{jim})R^m_k - \\ & - (C_{jkm} - C_{kjm})R^m_i - 2(C_{im}R^m_{jk} + \\ & + C_{jm}R^m_{ki} + C_{km}R^m_{ij}). \end{aligned}$$

Further transvection of (2.13) by y^j yields, with (1.2) (c) in mind,

$$(2.14) \quad 2(R_{ik} - F^2 R h_{ik}) = C_k^m R_{mi} - C_i^m R_{mk},$$

where $R_{ik} := R_{0i0k}$.

Moreover, if we use the hypothesis (2.9), then from (2.11) we have

$$(2.15) \quad R_{hj} = K_{hj}/(1 + \lambda).$$

Consequently, substituting (2.15) and (2.9) into (2.14) and noting (2.12), we get

$$(2.16) \quad R_{hj} = F^2 R h_{hj}.$$

Using (2.16), we can easily see that

$$(2.17) \quad (C_{ikm} + 2C_{kim})R^m_j - (C_{ijm} + 2C_{jim})R^m_k - (C_{jkm} - C_{kjm})R^m_i = 0,$$

because of $C_{ijk} = C_{kji}$.

On the other hand, we see

$$K_{h0jk} = -(g_{hm} + C_{hm})R^m_{jk} = -(1 + \lambda)R_{0hjk},$$

which yields

$$(2.18) \quad C_{im}R^m_{jk} + i|j|k = -\lambda K_{i0jk}/(1 + \lambda) + i|j|k = 0.$$

Therefore, if we apply (2.17) and (2.18) to the right hand side of (2.13), then we can conclude $r_{0ijk} = 0$, namely $R_{0ijk} = R(y_j g_{ik} - y_k g_{ij})$, which is equivalent to (2.10). Q.E.D.

Propositions 2.2 and 2.3 yield the following

Theorem 2.4. *If $C_{ij} = \lambda h_{ij}$ ($\lambda \neq -1$), the generalized metric space of R -isotropic curvature is characterized by*

$$R_h^i{}_{jk} = R(g_{hj}\delta_k^i - g_{hk}\delta_j^i).$$

§3. A generalized metric space of K -isotropic curvature

Secondly, we consider a generalized metric space of K -isotropic curvature. In this case, if we put

$$k_{hijk} := K_{hijk} - K(g_{hj}g_{ik} - g_{hk}g_{ij}),$$

then we must have

$$(3.1) \quad k_{hijk} + k_{jihk} + k_{hkji} + k_{jkhi} = 0.$$

It is easily shown that the tensor k_{hijk} satisfies the following relations:

$$(3.2) \quad k_{hijk} = -k_{hikj},$$

$$(3.3) \quad k_{hijk} - k_{jkhi} = K_{hijk} - K_{jkhi} \\ = \frac{1}{2}(T_{hijk} - g_{hi(m)}R^m{}_{jk} + g_{jk(m)}R^m{}_{hi}),$$

$$(3.4) \quad k_{hijk} + h[j|k = K_{hijk} + h[j|k = 0.$$

Consequently, k_{hijk} satisfies the conditions for A_{hijk} in Lemma 1.1, where $U_{hijk} = \frac{1}{2}(T_{hijk} - g_{hi(m)}R^m{}_{jk} + g_{jk(m)}R^m{}_{hi})$ and $V_{hijk} = 0$. Therefore k_{hijk} has the form

$$6k_{hijk} = \frac{1}{2}(T_{hijk} + T_{jihk} - T_{kijh} - T_{hikj} - g_{hi(m)}R^m{}_{jk} - g_{ji(m)}R^m{}_{hk} + \\ + g_{ki(m)}R^m{}_{hj} + g_{hi(m)}R^m{}_{kj} + g_{jk(m)}R^m{}_{hi} + g_{hk(m)}R^m{}_{ji} - \\ - g_{hj(m)}R^m{}_{ki} - g_{kj(m)}R^m{}_{hi}).$$

Making use of the definition of T_{hijk} and (1.15) in the above equation, we obtain

$$(3.5) \quad 6k_{hijk} = (2g_{hj(m)}R^m{}_{ik} + g_{ik(m)}R^m{}_{hj} - j|k) - 2g_{hi(m)}R^m{}_{jk}.$$

From this equation, we shall derive the following interesting result.

Theorem 3.1. *A generalized metric space of non vanishing K -isotropic curvature is a Riemannian space of constant curvature.*

PROOF. Interchanging indices h and j in (3.3), we get

$$(3.3)' \quad k_{jihk} - k_{hkji} = \frac{1}{2}(T_{jihk} - g_{ji(m)}R^m_{hk} + g_{hk(m)}R^m_{ji}).$$

Summing the three equations (3.1), (3.3) and (3.3)', we have

$$(3.6) \quad 2(k_{hijk} + k_{jihk}) = \frac{1}{2}(T_{hijk} + T_{jihk} - g_{hi(m)}R^m_{jk} + g_{jk(m)}R^m_{hi} - g_{ji(m)}R^m_{hk} + g_{hk(m)}R^m_{ji}).$$

If we consider $+h|j|k$ in (3.6) and use (3.4) and (1.15), we obtain

$$(3.7) \quad g_{hj(m)}R^m_{ki} + h|j|k = 0.$$

Transvecting (3.7) by y^h and making use of (1.2) (b), we have

$$(3.8) \quad C_{jm}R^m_{ki} + g_{jk(m)}R^m_i + C_{km}R^m_{ji} = 0.$$

Further transvection of (3.8) by y^k yields, with (1.2) (c) in mind,

$$(3.9) \quad C_{jm}R^m_i = 0.$$

Transvecting (3.5) by $y^h y^j$ and using (3.9), we obtain

$$(3.10) \quad k_{0i0k} = 0,$$

which gives, by the definition of k_{hijk} ,

$$(3.11) \quad R_{ik} = F^2 K h_{ik}.$$

Substituting (3.11) into (3.9), we have

$$(3.12) \quad F^2 K C_{jk} = 0.$$

Since we assume $K \neq 0$, we must have $C_{jk} = 0$. From this result, (3.8) and (3.11), we obtain $g_{jk(i)} = 0$. This means that the space in consideration is a Riemannian space. Q.E.D.

The above proof also yields the following

Corollary 3.2. *A Finsler space of K -isotropic curvature is a Riemannian space of constant curvature.*

§4. A generalized metric space of H -isotropic curvature

Thirdly, we consider a generalized metric space of H -isotropic curvature. In this case, if we put

$$h_{hijk} := H_{hijk} - H(g_{hj}g_{ik} - g_{hk}g_{ij}),$$

then we must have

$$(4.1) \quad h_{hijk} + h_{jihk} + h_{hkji} + h_{jkhi} = 0.$$

On the other hand, the tensor h_{hijk} obeys the following relations:

$$(4.2) \quad h_{hijk} = -h_{hikj},$$

$$(4.3) \quad h_{hijk} - h_{jkhi} = H_{hijk} - H_{jkhi}$$

$$= \frac{1}{2}(T_{hijk} - g_{hi(m)}R^m_{jk} + g_{jk(m)}R^m_{hi}) + E_{hijk} - E_{jkhi},$$

$$(4.4) \quad h_{hijk} + h|j|k = H_{hijk} + h|j|k = 0.$$

Consequently, h_{hijk} satisfies the conditions for A_{hijk} in Lemma 1.1, where $U_{hijk} = \frac{1}{2}(T_{hijk} - g_{hi(m)}R^m_{jk} + g_{jk(m)}R^m_{hi}) + E_{hijk} - E_{jkhi}$, $V_{hijk} = 0$. Therefore h_{hijk} has the form

$$\begin{aligned} 6h_{hijk} = & \frac{1}{2}(T_{hijk} + T_{jihk} - T_{kijh} - T_{hikj} - g_{hi(m)}R^m_{jk} - g_{ji(m)}R^m_{hk} + \\ & + g_{ki(m)}R^m_{hj} + g_{hi(m)}R^m_{kj} + g_{jk(m)}R^m_{hi} + g_{hk(m)}R^m_{ji} - \\ & - g_{hj(m)}R^m_{ki} - g_{kj(m)}R^m_{hi}) + E_{hijk} - E_{jkhi} + E_{jihk} - E_{hkji} - \\ & - E_{kijh} + E_{hjki} - E_{hikj} + E_{kjhi}. \end{aligned}$$

Making use of the definition of T_{hijk} , (1.15) and (1.10) (d), we obtain

$$(4.5) \quad \begin{aligned} 6h_{hijk} = & (2g_{hj(m)}R^m_{ik} + g_{ik(m)}R^m_{hj} - j|k) - 2g_{hi(m)}R^m_{jk} + \\ & + 3E_{hijk} - E_{jkhi} + E_{kjhi} - E_{hkji} + E_{hjki}. \end{aligned}$$

Now we put $E_{ik} := E_{0i0k}$, $\bar{E}_{hj} := E_{h0j0}$, and we first show two lemmas in a generalized metric space.

Lemma 4.1. *In a generalized metric space, we have*

$$(4.6) \quad \bar{E}_{hj} = \bar{E}_{jh}.$$

PROOF. Transvecting (1.10) (d) by $y^i y^k$ and using (1.9) (e), we have
(4.6). Q.E.D.

Lemma 4.2. *In a generalized metric space, we have*

$$(4.7) \quad \bar{E}_{hj} = E_{hj} + C_{hi}E^i_j.$$

PROOF. Using (1.5), i.e.

$$(4.8) \quad C_{hj/0} = -2D_{h0j} = 2(g_{ij} + C_{ij})P^i_h$$

and (1.9) (e), we get

$$\begin{aligned} \bar{E}_{hj} &= D_{h0j/0} - P^i_h D_{i0j} = -(g_{ij} + C_{ij})P^i_{h/0} - C_{ij/0}P^i_h - D_{i0j}P^i_h \\ &= -(g_{ij} + C_{ij})P^i_{h/0} + D_{m0j}P^m_h = -(g_{ij} + C_{ij})(P^i_{h/0} + P^i_m P^m_h) \\ &= (g_{ij} + C_{ij})E^i_h = E_{jh} + C_{ij}E^i_h = E_{hj} + C_{hi}E^i_j. \end{aligned}$$

Q.E.D.

Next, we consider a generalized metric space of H -isotropic curvature.

Lemma 4.3. *In a generalized metric space of H -isotropic curvature, we have*

$$(4.9) \quad E_{ji} = \bar{E}_{ji} + C_{jm}R^m_i.$$

PROOF. Interchanging indices h and j in (4.3), we get

$$(4.3)' \quad h_{jihk} - h_{hkji} = \frac{1}{2}(T_{jihk} - g_{ji(m)}R^m_{hk} + g_{hk(m)}R^m_{ji}) + E_{jihk} - E_{hkji}.$$

Adding (4.1), (4.3) and (4.3)', we obtain

$$\begin{aligned} 2(h_{hijk} - h_{jikh}) &= \frac{1}{2}(T_{hijk} + T_{jihk} - g_{hi(m)}R^m_{jk} + \\ (4.10) \quad &+ g_{jk(m)}R^m_{hi} - g_{ji(m)}R^m_{hk} + g_{hk(m)}R^m_{ji}) + \\ &+ E_{hijk} - E_{jkhi} + E_{jihk} - E_{hkji}. \end{aligned}$$

Considering $+h|j|k$ in (4.10) and noticing (4.4) and (1.10) (d), we have

$$(4.11) \quad g_{hk(m)}R^m_{ij} + (E_{jkhi} + E_{hkji}) + h|j|k = 0.$$

Transvecting (4.11) by y^h , we have

$$\begin{aligned} (4.12) \quad &C_{km}R^m_{ij} + C_{jm}R^m_{ik} - g_{jk(m)}R^m_i + E_{jk0i} + E_{j0ki} + E_{k0ji} + \\ &+ E_{kj0i} + E_{0jki} + E_{0kji} = 0. \end{aligned}$$

Moreover, transvecting (4.12) by y^k and using (1.9) (e), we have (4.9).
Q.E.D.

Lemma 4.4. *In a generalized metric space of H -isotropic curvature, we have*

$$(4.13) \quad H^i_k = F^2 H h^i_k.$$

PROOF. Transvecting (4.5) by $y^h y^j$, we see that

$$6h_{0i0k} = 6(H_{ik} - F^2 H h_{ik}) = 2C_{km} R^m_i - 3C_{im} R^m_k + 3E_{ik} - 2E_{ki} - \bar{E}_{ki}.$$

Lemmas 4.1 and 4.3 tell us that the right hand side of the above equation vanishes and then (4.13) holds. Q.E.D.

Lemma 4.5. *A generalized metric space of H -isotropic curvature is a Finsler space.*

PROOF. From Lemmas 4.2 and 4.3, we see

$$C_{hi} E^i_j = -C_{hi} R^i_j.$$

Hence, using (1.9) (d) and Lemma 4.4, we get

$$0 = C_{hi} (R^i_j + E^i_j) = C_{hi} H^i_j = F^2 H C_{hi} h^i_j = F^2 H C_{hj},$$

from which we have $C_{hj} = 0$. Therefore the space in consideration reduces to a Finsler space. Q.E.D.

Lemma 4.6. *In a generalized metric space of H -isotropic curvature, we have*

$$(4.14) \quad \begin{aligned} (a) \quad & E_h^i{}_{jk} = P_h^i{}_{j/k} + P_h^m{}_j P_m^i{}_k - j|k, \quad (b) \quad E^i{}_{jk} = 0, \quad E^i{}_k = 0, \\ (c) \quad & E_h^i{}_{j0} = P_h^i{}_{j/0}, \quad E_h^i{}_{0k} = -P_h^i{}_{k/0}. \end{aligned}$$

PROOF. From Lemma 4.5, the space in consideration is a Finsler space. Then, noticing (4.8) we have $P^i{}_k = 0$, which means that $D_j^i{}_k = P_j^i{}_k + P^i{}_{j(k)} = P_j^i{}_k$. Hence (4.14) (a) follows from (1.9) (b) and the other from $P_0^i{}_k = 2P^i{}_k = 0$. Q.E.D.

Now we ready to prove the following

Theorem 4.7. *A generalized metric space of H -isotropic curvature is a Finsler space of constant curvature.*

PROOF. Transvecting (4.5) by y^h and using Lemma 4.6, we have

$$6h_{0ijk} = -2C_{ijm} R^m_k + 2C_{ikm} R^m_j + 3E_{ijk} - E_{jk0i} + E_{kj0i} - E_{kji} + E_{jki}.$$

Noting $R^i{}_k = H^i{}_k + F^2 H h^i_k$ and Lemma 4.6, we obtain

$$(4.15) \quad 6h_{0ijk} = 6(H_{ijk} - H(y_j g_{ik} - y_k g_{ij})) = 0, \text{ or } H^i{}_{jk} = H(y_j \delta_k^i - y_k \delta_j^i).$$

By virtue of a well-known theorem ([4], p.133), a Finsler space which satisfies (4.15) is a Finsler space of constant curvature, that is,

$$h_{hijk} = H_{hijk} - H(g_{hj} g_{ik} - g_{hk} g_{ij}) = 0.$$

Q.E.D.

Remark. The right hand side of (4.5) reduces to $2(P_{hij/k} + HC_{hij}y_k) - j|k$ after some calculation using (4.15). This is consistent with the well-known identity in a Finsler space (e.g. [2], (2.7) (b))

$$H_{hijk} + H_{ihjk} = 2(P_{hij/k} - P_{hik/j}) - 2C_{him}H_j^m{}_k.$$

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