

On extrinsic spheres in Kähler manifolds

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1. Introduction

An m -dimensional submanifold, $m \geq 2$, of a Riemannian manifold is called to be an extrinsic sphere if it is totally umbilical and its mean curvature vector field is non-zero and parallel. Extrinsic spheres have been geometrically characterized by NOMIZU and YANO [2]. Since they have the same extrinsic properties as ordinary spheres in a Euclidean space, it is natural to ask when an extrinsic sphere is isometric to an ordinary sphere.

When the ambient manifold is a Kähler manifold, it is proved that:

Theorem A [1]. *A complete, connected, simply connected and even-dimensional extrinsic sphere of a Kähler manifold is isometric to an ordinary sphere if its normal connection is flat.*

Theorem B [4]. *A complete, connected and simply connected extrinsic sphere M in a Kähler manifold $\tilde{M}$ is one of the following:*

- (1) M is isometric to an ordinary sphere,
- (2) M is homothetic to a Sasakian manifold,
- (3) M is a totally real submanifold and the f -structure is not parallel in the normal bundle.

Here we note that the cases (2) and (3) occur only when $\dim M = m$ is odd and $\dim \tilde{M} \geq m + 1$, respectively.

The main theorem of the present paper is as follows:

Theorem 1.1. *Let $\tilde{M}$ be a Kähler manifold and M a complete, connected and simply connected extrinsic sphere of dimension $m \geq 3$ in $\tilde{M}$. Assume, additionally, that the curvature tensor of the connexion induced in the normal bundle of M satisfies the condition*

$$(1.1) \quad \nabla_n \nabla_m R_{jiy}x - \nabla_m \nabla_n R_{jiy}x = 0.$$

Then one of the following cases occurs:

- 1) M is isometric to an ordinary sphere,
- 2) M is homothetic to a Sasakian manifold.

Moreover, we give some sufficient conditions for an extrinsic sphere to be homothetic to a Sasakian manifold. In the proofs we use the following well-known theorem of Okumura [3].

Theorem C. *Let M be a Riemannian manifold. If M admits a Killing vector field v of constant length satisfying*

$$\nabla_k \nabla_j v_i = h^2 (v_j g_{ik} - v_i g_{jk}),$$

for a non-zero constant h , then M is homothetic to a Sasakian manifold.

2. Preliminaries

Let $\tilde{M}$ be a Kähler manifold of dimension n and (F, G) its Kähler structure. Then we have

$$(2.1) \quad F^\alpha{}_\gamma F^\gamma{}_\beta = -\delta^\alpha_\beta, \quad F^\lambda{}_\alpha F^\gamma{}_\beta G_{\lambda\gamma} = G_{\alpha\beta}, \quad \nabla_\gamma F^\alpha{}_\beta = 0.$$

Here and in the sequel the indices $\alpha, \beta, \dots$ run over the range $\{1, 2, \dots, n\}$ and ∇ denotes the covariant differentiation in $\tilde{M}$.

Let M ($\dim M = m$, $m < n$) be a submanifold of $\tilde{M}$. Denote by (x^α) and (u^i) local coordinates of $\tilde{M}$ and M , respectively. The continual indices $h, i, j, \dots$ run over the range $\{1, 2, \dots, m\}$.

Let $x^\alpha = x^\alpha(u^i)$ be a local parametric representation of M in $\tilde{M}$. Set $B_i^\alpha = \partial x^\alpha / \partial u^i$. M inherits from $\tilde{M}$ the Riemannian metric g of local components $g_{ij} = G_{\alpha\beta} B_i^\alpha B_j^\beta$. Next we take $n - m$ mutually orthogonal unit local vector fields normal to M and denote their local components by N_x^α . The indices $x, y, z, u, w, \dots$ run over the range $\{m+1, \dots, n\}$. Denote by g_{xy} the components of the metric tensor induced on the normal bundle $T^\perp M$ of M from the metric tensor G of $\tilde{M}$, that is $g_{xy} = G_{\alpha\beta} N_x^\alpha N_y^\beta$.

Let us express FB_i and FN_x as linear combinations of B_i and N_x as follows

$$(2.2) \quad F_\gamma{}^\alpha B_i{}^\gamma = f_i{}^r B_r{}^\alpha + f_i{}^x N_x{}^\alpha,$$

$$(2.3) \quad F_\gamma{}^\alpha N_x{}^\gamma = f_x{}^r B_r{}^\alpha + f_x{}^y N_y{}^\alpha.$$

Then $f_y{}^x$ are components of an f -structure in the normal bundle $T^\perp M$. It can easily be noted that

$$f_{ij} = -f_{ji}, \quad f_{ix} = -f_{xi}, \quad f_{xy} = -f_{yx},$$

where

$$f_{ij} = f_i{}^r g_{rj}, \quad f_{ix} = f_i{}^y g_{xy}, \quad f_{xi} = f_x{}^r g_{ri}, \quad f_{xy} = f_x{}^y g_{vy}.$$

With the help of (2.2) - (2.3), we find

$$(2.4) \quad f_j^r f_{ri} + f_j^x f_{xi} = -g_{ij},$$

$$(2.5) \quad f_i^r f_{rx} + f_i^v f_{vx} = 0,$$

$$(2.6) \quad f_x^v f_{vy} + f_x^r f_{ry} = -g_{xy}.$$

If $FT_x M \subset T_x^\perp M$ for any $x \in M$ (i.e. $f_j^i = 0$), then M is said to be a totally real submanifold of $\tilde{M}$.

Denoting by h_{ij}^x the components of the second fundamental tensor of the submanifold M , we have the following equations of Gauss

$$(2.7) \quad \nabla_j B_i^\alpha = h_{ij}^v N_v^\alpha,$$

and of Weingarten

$$(2.8) \quad \nabla_j N_x^\alpha = -h_j^r B_r^\alpha, \quad h_j^r B_r^\alpha = h_{ji}^v g^{ir} g_{vx},$$

where ∇ denotes the Van der Waerden - Bortolotti covariant differentiation (see e.g. YANO and ISIHARA [5]).

The mean curvature vector field of the submanifold M has local components $h^x = 1/mg^{rs}h_{rs}^x$ and the function h is such that $h^2 = g_{xy}h^xh^y$ is the mean curvature.

The submanifold M is said to be totally umbilical if $h_{ij}^x = g_{ij}h^x$ and totally geodesic if $h_{ij}^x = 0$. If the mean curvature vector field of a totally umbilical submanifold is non - zero and parallel ($\nabla_j h^x = 0$), then M is called an extrinsic sphere in $\tilde{M}$ (cf. [2]). The mean curvature h of an extrinsic sphere is a non - zero constant.

Let M be an extrinsic sphere in $\tilde{M}$. Differentiating (2.2) and (2.3) covariantly along the submanifold M we obtain, by virtue of (2.1) - (2.3), (2.7) and (2.8),

$$(2.9) \quad \nabla_k f_{ij} = v_i g_{jk} - v_j g_{ik},$$

$$(2.10) \quad \nabla_k f_{ix} = f_{vx} h^v g_{ik} - h_x f_{ik},$$

$$(2.11) \quad \nabla_k f_{xy} = h_y f_{kx} - h_x f_{ky},$$

where $v_i = f_{ix} h^x$.

The covector field $v = (v_i)$ satisfies

$$(2.12) \quad \nabla_j v_i = -h^2 f_{ij}.$$

Thus it is Killing. Differentiating again (2.12) covariantly and making use of (2.9), we obtain

$$(2.13) \quad \nabla_k \nabla_j v_i = h^2 (v_j g_{ik} - v_i g_{jk}).$$

3. Extrinsic spheres

Lemma 3.1. *For an extrinsic sphere in a Kähler manifold, the following relations hold*

$$(3.1) \quad v^r T_{kji}^r = 0,$$

$$(3.2) \quad f_{rx} T_{kji}^r + f_i^v R_{k j x v} = 0,$$

$$(3.3) \quad \begin{aligned} & f_{rx} (\nabla_n \nabla_m R_{kji}^r - \nabla_m \nabla_n R_{kji}^r) + \\ & f_{iv} (\nabla_n \nabla_m R_{k j x}^v - \nabla_m \nabla_n R_{k j x}^v) = \\ & = h^2 (f_{nx} T_{k j i m} - f_{mx} T_{k j i n} + g_{in} f_{rx} T_{k j m}^r - g_{im} f_{rx} T_{k j n}^r), \end{aligned}$$

where

$$(3.4) \quad T_{k j i h} = R_{k j i h} - h^2 (g_{kh} g_{ij} - g_{ki} g_{jh}),$$

while $R_{k j i h}$ and $R_{k j x y} = R_{k j}^v g_{vy}$ are covariant components of the curvature tensor of M and the curvature tensor of the connexion induced in the normal bundle of M , respectively.

PROOF. (3.1) follows from (2.13), the Ricci identity and (3.4). Differentiating now (2.10) covariantly and applying (2.9), (2.11) and the Ricci identity, we obtain

$$f_{rx} R_{kji}^r + f_i^v R_{k j x v} = h^2 (g_{ij} f_{kx} - g_{ik} f_{jx}).$$

This, by making use of (3.4), can be written in the form (3.2). To prove (3.3) we transvect the Ricci identity

$$\begin{aligned} \nabla_n \nabla_m R_{j i y x} - \nabla_m \nabla_n R_{j i y x} = & -R_{ri y x} R_{n m y}^r - R_{j r y x} R_{n m i}^r - \\ & - R_{j i v x} R_{n m y}^r - R_{j i y v} R_{n m x}^v, \end{aligned}$$

with f_k^x and use (3.2), (3.4) and the known identity

$$\begin{aligned} \nabla_n \nabla_m R_{h k j i} - \nabla_m \nabla_n R_{h k j i} = & -R_{r k j i} R_{n m h}^r - R_{h r j i} R_{n m k}^r - \\ & - R_{h k r i} R_{n m j}^r - R_{h k j r} R_{n m i}^r. \end{aligned}$$

This completes the proof.

The above lemma will be used in the following proof:

PROOF OF THEOREM 1.1. Suppose that M is a complete, connected and simply connected extrinsic sphere in a Kähler manifold M .

By Theorem B, it is sufficient to show that in the case (3) of Theorem B M is isometric to an ordinary sphere if its curvature tensor of the normal

connexion satisfies (1.1). Let M satisfy (1.1) and be totally real ($f_{ij} = 0$). Then, from (3.3) it follows that

$$(3.5) \quad f_{rx}(\nabla_n \nabla_m R_{kji}{}^r - \nabla_m \nabla_n R_{kji}{}^r) = h^2(f_{nx}T_{kjim} - f_{mx}T_{kjin} + \\ + g_{in}f_{rx}T_{kjm}{}^r - g_{im}f_{rx}T_{kjn}{}^r),$$

and from (2.4)

$$(3.6) \quad f_j{}^x f_{xi} = -g_{ji}.$$

Next, permuting (3.5) cyclically with respect to the indices (k, j, i) , adding the resulting equations and using the relation $T_{kji}{}^r + T_{jik}{}^r + T_{ikj}{}^r = 0$ and the first Bianchi identity, we find

$$f_x{}^r(g_{in}T_{kjm}{}^r - g_{im}T_{kjn}{}^r + g_{kn}T_{jim}{}^r - g_{km}T_{jin}{}^r + g_{jn}T_{ikm}{}^r - g_{jm}T_{ikn}{}^r) = 0.$$

Hence, by transvection with $f_h{}^x g^{im}$ and making use of (3.6), we have

$$(3.7) \quad (m-3)T_{kjni}{}^h = T_{jh}g_{kn} - T_{kh}g_{jn},$$

where $T_{kh} = T_{kjni}{}^h g^{jn}$, which by transvection with g^{jn} yields $T_{kh} = 0$ for $m \geq 3$. If $m > 3$, then the relation $T_{kh} = 0$ used in (3.7) yields $T_{kji}{}^h = 0$. As it is known, if $m = 3$, then $T_{kh} = 0$ always implies $T_{kji}{}^h = 0$. Thus, M is a manifold of constant curvature. As M is complete, connected and simply connected, M is isometric to an ordinary sphere. The proof is complete.

If $\nabla_k f_y{}^x = 0$, then the f -structure in the normal bundle $T^\perp M$ is said to be parallel. It is known that if M is a totally real submanifold, then the f -structure is not parallel (cf. [4], Theorem B). For an f -structure which is parallel, we have the following.

Theorem 3.2. *Let M be an extrinsic sphere in a Kähler manifold. If the f -structure in $T^\perp M$ is parallel, then M is homothetic to a Sasakian manifold.*

PROOF. Suppose that $\nabla_k f_y{}^x = 0$ holds. Then, from (2.11) it follows that $f_{kx}h_y - f_{ky}h_x = 0$. Hence, by covariant differentiation and using (2.10), we have

$$(3.8) \quad h_y f_{zx} h^z - h_x f_{zy} h^z = 0.$$

Now, transvecting (3.8) with h^y , we get $f_{zx}h^z = 0$. On the other hand, by transvection of (2.6) with $h^x h^y$, we obtain, in virtue of the above relation, $v_r v^r = h^2$. The vector field v is Killing (see 2.12), satisfies (2.13) and has non-zero constant length. But if a Riemannian manifold admits such a vector field, then it is homothetic to a Sasakian manifold [3]. The proof is complete.

Corollary 3.3. *If the f -structure in the normal bundle of M vanishes, then M is a hypersurface homothetic to a Sasakian manifold.*

PROOF. By Theorem 3.2, it is sufficient to show that M is a hypersurface. Suppose that $f_{yx} = 0$ holds on M .

Then, (2.11) yields $f_{kx}h_y - f_{ky}h_x = 0$, which, by transvection with $f_z^k h^y$ and the use of (2.6) gives $h^2 g_{zx} = h_z h_x$ at each point of M . From this we conclude that $\dim M = n - 1$, which completes the proof.

Finally, note that from (2.9) and (2.11) we can easily obtain the following.

Proposition 3.4. *An extrinsic sphere M in a Kähler manifold is totally real if and only if the tensor field of components f_{ij} is parallel on M .*

Contrary to the above we have

Proposition 3.5. *Let M be an extrinsic sphere in a Kähler manifold. There are no open subsets of M on which $\nabla_k f_{jx} = 0$.*

PROOF. Suppose that U is an open subset of M on which $\nabla_k f_{jx} = 0$. Then, from (2.10) it follows that $f_{ij} = 0$ and $f_{zx}h^z = 0$ on M . From (2.9) we get $v_i = 0$ and from (2.11) $h^2 f_{jx} = v_j h_x$. Consequently, we have $f_{jx} = 0$ on U . This contradicts the equality (2.4). The proof is complete.

References

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(Received Januar 3, 1989)