

On k -Lagrange geometry

By MAGDALEN SZ. KIRKOVITS (Sopron)

Abstract. In this paper we describe the theory of k -Lagrange-geometry. It is a suitable geometrical model for studying variational problems of multiple integrals in a geometrical manner. We consider the vector bundle $\eta = \left(\bigoplus_1^k TM, \pi, M \right)$ and study the geometry of the total space $E = \bigoplus_1^k TM$.

1. Introduction

Preliminaires and motivations.

It is known that the fundamental problem in the calculus of variations for multiple integrals can be briefly formulated as follows ([11]). Let R be some domain of the space of the variables x^i . (In the sequel Latin and Greek indices assume the values $1, \dots, n$ and $1, \dots, k$ ($k < n$), respectively, the summation convention being used in both cases.) We will call *admissible* the class of functions $x^i(t^\alpha), \bar{x}^i(t^\alpha), \dots$ defined on the same domain G_t of t^α if they are of class C^2 and coincide with each other on the boundary ∂G_t of G_t .

Suppose that we have a function $\mathcal{L}(x^j(t^\beta), \dot{x}_\alpha^j(t^\beta))$; $\dot{x}_\alpha^j := \partial x^j / \partial t^\alpha$, also of class C^2 and defined over each space $C_k : x^i = x^i(t^\alpha)$ of the admissible class. Moreover, let G_t be a fixed, bounded and simply connected domain in the k -dim. space of t^α . One can then form the following k -fold integral

$$(1.1) \quad I(C_k) = \int_{G_t} \mathcal{L}(x^j(t^\beta), \dot{x}_\alpha^j(t^\beta)) d(t); \quad d(t) := dt^1 \dots dt^k.$$

The fundamental variational problem for a multiple integral (1.1) is to establish necessary and sufficient conditions for an admissible set of functions $x^i(t^\alpha)$ in order that it gives an extreme of (1.1) relative to other

admissible sets. A necessary condition for this is that the first variation δI of the fundamental integral (1.1) should vanish. This implies that $x^i(t^\alpha)$ must satisfy the system of n second order partial differential equations:

$$(1.2) \quad \varepsilon_i(\mathcal{L}) := \frac{d}{dt^\alpha} \frac{\partial \mathcal{L}}{\partial \dot{x}_\alpha^i} - \frac{\partial \mathcal{L}}{\partial x^i} = 0 \quad (\text{summation over } \alpha).$$

Remark. ε_i are the components of the covariant Euler–Lagrange vector ([12]).

A question of the variational calculus for single or multiple integrals is the *equivalence* between two variational problems of the same type. This was studied by C. CARATHÉODORY ([1]), A. MOÓR ([9], [10]) and also by H. RUND ([11], [12]).

In [4] the authoress has considered a generalized version of the equivalence of two variational problems for single integrals treated by MOÓR ([9]). This problem has the following form in Lagrange spaces $(M, \mathcal{L}^*(x, y))$ and $(M, \mathcal{L}(x, y))$ ([6], [7])

$$(1.3) \quad \varepsilon_i(\mathcal{L}^*(x, y)) = \lambda(x, y) \varepsilon_i(\mathcal{L}(x, y)); \quad \lambda(x, y) \neq 0,$$

where λ depends not only on x but on y too. In [4] two geometrical conditions were found which are equivalent to (1.3). Moreover, necessary and sufficient conditions for this equivalence were established. We note that in the proofs *only geometrical methods* of the theory of Lagrange spaces were used.

A. MOÓR ([10]) gave the most general definition of the *equivalence of two variational problems for multiple integrals* and he investigated it in some important cases but he did not investigate its geometrical meanings.

Our purpose is to construct a *geometrical model* for multiple integrals in the calculus of variations, then to study the MOÓR equivalence in a geometrical manner and to give other applications. In a joint paper ([8]) we have briefly sketched the first results. Now we describe the theory of k -Lagrange geometry by using as a model the geometry of the total space of a vector bundle developed by R. MIRON ([5]). We remark that our theory is based on the study of a metric which is derived from a Lagrangian and thus it differs from Günther's theory ([3]).

2. Vector bundles, differential structure on $E = \bigoplus_1^k TM$

We consider the 1-jet bundle $\mathcal{J}^1(R^k, TM)$ ($k < n$) over an n -dimensional manifold M . This bundle is isomorphic to the vector bundle $\text{Hom}(R^k, TM) \rightarrow M$. Moreover, if we fix a basis $(e_1, \dots, e_k)$ of R^k there is

the isomorphism: $\text{Hom}(R^k, TM) \simeq \bigoplus_1^k TM = \overbrace{TM \oplus TM \oplus \dots \oplus TM}^{k\text{-times}}$ ([3]).

We shall systematically use the latter fact. Denoting $E = \bigoplus_1^k TM$ and by π its projection on M , we shall study the vector bundle $\eta = (E, \pi, M)$ and the geometry of the total space E . Clearly $\dim E = nk$.

Let (U, ψ) be a chart on M . Then $(U \times R^{kn}, \varphi)$ is a vector chart of the vector bundle η where $\varphi : \pi^{-1}(U) \rightarrow U \times R^{kn}$. For any $X_x \in \bigoplus_1^k T_x M$, $x \in M$ we get

$$(2.1) \quad \varphi(X_x) = (y_\alpha^i), \quad y_\alpha^i \in T_x^\alpha M.$$

Hence we have for every fixed α

$$(2.2) \quad X_\alpha^i = y_\alpha^i (\partial / \partial x^i)_x.$$

This means that any vector $X_x \in \bigoplus_1^k T_x M$ is determined by the following components

$$(2.3) \quad X_x = (y_1^i \partial / \partial x^i, \dots, y_k^i \partial / \partial x^i).$$

We put $(x^i) = \psi(x)$ and define

$$(2.4) \quad h : \pi^{-1}(U) \rightarrow \psi(U) \times R^{kn}$$

by

$$(2.5) \quad h(X_1^i, \dots, X_k^i) = (x^i, y_\alpha^i) \in R^n \times R^{kn}$$

which are *canonical coordinates* on $\pi^{-1}(U)$. The set of charts $(\pi^{-1}(U), h)$ defines a vector atlas on $E = \bigoplus_1^k TM$.

Denote $\psi_j \circ \psi_i^{-1}(x^s) = (\bar{x}^s)$, then

$$(h_j \circ h_i^{-1})(x^s, y_\alpha^s) = (\bar{x}^s(x^1, \dots, x^n), \partial_k \bar{x}^s(x) y_\alpha^k) \quad (\partial_k := \partial / \partial x^k)$$

i.e. for a coordinate transformation on U the corresponding coordinate transformation on $\pi^{-1}(U)$ is

$$(2.6) \quad \begin{aligned} \bar{x}^s &= \bar{x}^s(x^1, \dots, x^n); \quad \text{rank}(\partial_k \bar{x}^s) = n \\ \bar{y}_\alpha^s &= \partial_k \bar{x}^s y_\alpha^k \quad (\alpha = \overline{1, k}). \end{aligned}$$

The transformation law shows that (y_α^i) can be considered as a *contravariant vector*. In the sequel we denote y_α^i by y^a where $\binom{i}{\alpha} := a$ and use a shorter notation $a, b, c, \dots (a', b', c', \dots)$ instead of double contravariant (or covariant) indices $\binom{i}{\alpha}$ (or $\binom{\alpha}{i}$) and $\binom{i'}{\alpha}$ (or $\binom{\alpha}{i'}$) respectively, if the computation allows it.

Let $T_u E$ be the tangent space of E at u . Its basis is $(\partial_i, \partial_i^\alpha) := (\partial_i, \partial_a)$, where $\partial_i := \partial/\partial x^i$ and $\partial_a := \partial_i^\alpha = \partial/\partial y_\alpha^i$. Hence a tangent vector $X_u \in T_u E$ looks locally as follows

$$(2.7) \quad \begin{aligned} X_u &= X^i \partial_i + \dot{X}_\alpha^i \partial_i^\alpha := X^i \partial_i + \dot{X}^a \partial_a \\ (a = \overline{1, nk}, (X^i) \in R^n, (\dot{X}^a) \in R^{nk}). \end{aligned}$$

The change of the local basis on $E = \bigoplus_1^k TM$ is given by

$$(2.8) \quad \begin{aligned} \partial_i &= \partial_i \bar{x}^k \bar{\partial}_k + \partial_j \partial_i \bar{x}^k y_\alpha^j \bar{\partial}_k^\alpha; \quad (\bar{\partial}_k := \partial/\partial \bar{x}^k) \quad (\bar{\partial}_k^\alpha := \partial/\partial \bar{y}_\alpha^k) \\ \partial_i^\alpha &= \partial_i \bar{x}^k \bar{\partial}_k^\alpha \quad (\alpha = \overline{1, k}). \end{aligned}$$

The dual basis is denoted by (dx^i, dy_α^i) . Its transformation law follows from (2.6):

$$(2.9) \quad \begin{aligned} d\bar{x}^i &= \partial_k \bar{x}^i dx^k; \\ d\bar{y}_\alpha^i &= \partial_j \partial_k \bar{x}^i y_\alpha^j dx^k + \partial_k \bar{x}^i dy_\alpha^k, \quad (\alpha = \overline{1, k}). \end{aligned}$$

Let us consider the vector bundle $\left(\bigoplus_1^k TM, \pi, M \right) := (E, \pi, M)$. Then $D\pi : TE \rightarrow TM$ is the *differential* map of π . The mapping $D\pi$ is a π -morphism which maps the tangent bundle (TE, π_E, E) of E into the tangent bundle (TM, π, M) of M . Here $\pi_E : TE \rightarrow E$ and $\pi : TM \rightarrow M$ are the projections. Put $(VE, \pi_V, E) := \ker D\pi$. $VE = (VE, \pi_V, E)$ is called *vertical subbundle* over E . Its total space is $VE = \bigcup_{u \in E} V_u E$. The vertical subspace $V_u E$ of $T_u E$ is spanned by $\{\partial_a\}$. It is easy to see that

$X_u \in V_u E$ iff $X^i = 0$. The map $u \rightarrow V_u E$ over E determines the vertical distribution. Since $[\partial_a, \partial_b] = 0$ one gets that this distribution is integrable.

Now we define for each α an operator

$$(2.10) \quad \overset{\alpha}{\mathcal{J}} : T_u E \rightarrow T_u E$$

by

$$(2.11) \quad \overset{\alpha}{\mathcal{J}}(\partial_i) = \partial_i^\alpha, \quad \overset{\alpha}{\mathcal{J}}(\partial_i^\beta) = 0 \quad (\beta = \overline{1, k}).$$

It is easy to check that

$$(2.12) \quad \overset{\alpha}{\mathcal{J}}^2 = 0 \quad \text{and} \quad \ker \overset{\alpha}{\mathcal{J}} = \text{Im} \overset{\alpha}{\mathcal{J}} = V_u E$$

hold for every α .

So we have obtained that the manifold E can be endowed with k -different almost tangent structures.

It is not difficult to see that the Nijenhuis tensor associated to $\overset{\alpha}{\mathcal{J}}$ vanishes for every α , i.e. the almost tangent structures are integrable.

3. Nonlinear connection on $E = \bigoplus_1^k TM$

By the general theory the following sequence of vector bundles

$$(3.1) \quad 0 \rightarrow VE \xrightarrow[\iota]{C} TE \xrightarrow{\pi^!} \pi^* TM \rightarrow 0$$

is an exact sequence. Here $\pi^*(TM)$ is the pull-back of TM over E by π , ι is the inclusion map and $\pi^!(X)$ is given by $\pi^!(X) = (\pi_E(X), D\pi(X))$ where $\pi_E : TE \rightarrow E$ is the projection.

Definition 3.1. A nonlinear connection on E is a splitting at the left of the sequence (3.1), i.e. a map $C : TE \rightarrow VE$ such that $C \circ \iota = \text{id}|_{VE}$. The kernel $HE = (HE, \pi_H, E)$ of the morphism C is a subbundle of $TE \xrightarrow{\pi_E} E$ which will be called the horizontal bundle over E .

One gets for the total spaces

$$(3.2) \quad TE = HE \oplus VE \quad (\text{Whitney sum}).$$

Conversely, the existence of a subbundle HE of $TE \xrightarrow{\pi_E} E$ which satisfies (3.2) implies the existence of a morphism like C , i.e. a nonlinear connection on E .

Remark 3.1. By a general result there exist always nonlinear connections on E provided M is paracompact ([5]).

The exact sequence (3.1) looks locally as follows

$$\begin{aligned} 0 \rightarrow \psi(U) \times R^{kn} \times \{0\} \times R^{kn} \xrightarrow{\iota} \psi(U) \times R^{kn} \times R^n \times R^{kn} \rightarrow \\ \rightarrow \psi(U) \times R^{kn} \times R^n \rightarrow 0, \end{aligned}$$

where

$$\iota(x, y, 0, \dot{X}) = (x, y, 0, \dot{X}); \quad \pi!(x, y, X, \dot{X}) = (x, y, X).$$

Here $x := (x^i)$, $y := (y^a)$ and $X := (X^i)$, $\dot{X} := (\dot{X}^a)$.

The map C is described locally as

$$(x, y, X, \dot{X}) \rightarrow (x, y, 0, C_\varphi(x, y, X, \dot{X})),$$

where C_φ is a map linear in X and $\dot{X}$. The condition $C \circ \iota = \text{id}|_{VE}$ implies that

$$(3.3) \text{ (a)} \quad C_\varphi(x, y, X, \dot{X}) = \dot{X}_\alpha^i + N_{\alpha j}^i(x, y)X^j := \dot{X}^a + N_j^a X^j.$$

This shows that C can locally be written as

$$(3.3) \text{ (b)} \quad (x, y, X, \dot{X}) \rightarrow (x, y, 0, \dot{X}^a + N_j^a(x, y)X^j).$$

We have obtained a set of real functions N_j^a defined on the domain of the local charts of E . These functions determine a nonlinear connection N .

It is not difficult to check that if $\bar{N}_j^a$ is a similar set of functions on $\pi^{-1}(V)$ with $U \cap V \neq \emptyset$ then on $\pi^{-1}(U \cap V)$ we have the following transformation law

$$(3.4) \quad \bar{N}_{\alpha j}^i(\bar{x}, \bar{y})\partial_s \bar{x}^j = \partial_k \bar{x}^i N_{\alpha s}^k(x, y) - y_\alpha^k \partial_s \partial_k \bar{x}^i.$$

Conversely, a set of functions N_j^a which transform by (3.4) when the local chart is changed, defines a nonlinear connection on E .

In view of (3.2) the existence of a nonlinear connection implies the existence of a π -isomorphism between $HE \xrightarrow{\pi_H} E$ and $E \xrightarrow{\pi} M$. It follows that every tangent vector field Z on M determines a horizontal vector Z^h on E such that $D\pi(Z^h) = Z$. Z^h is the horizontal lift of Z . Taking $\delta_i = (\partial_i)^h$ one obtains a local basis of $H_u E$ ($u \in E$). Generally we have

$\delta_i = A_i^j \partial_j + B_{\alpha i}^j \partial_j^\alpha$. From $D\pi(\delta_i) = \partial_i$ and $C(\delta_i) = 0$ we get $A_i^j = \delta_i^j$ and $B_{\alpha i}^j = -N_{\alpha i}^j$, respectively. Thus we have

$$(3.5) \quad \delta_i = \partial_i - N_{\alpha i}^j \partial_j^\alpha := \partial_i - N_i^a \partial_a.$$

It is clear that (δ_i, ∂_a) is an *adapted basis* for the decomposition (3.2). Its dual basis is $(dx^i, \delta y^a)$, where

$$(3.6) \quad \delta y^a = dy^a + N^a_j dx^j.$$

Their transformation laws are

$$(3.7) \quad \begin{aligned} \text{(a)} \quad \delta_i &= \partial_i \bar{x}^s \bar{\delta}_s \quad (\bar{\delta}_s := \delta / \delta \bar{x}^s) & \text{(b)} \quad \partial_i^\alpha &= \partial_i \bar{x}^s \bar{\partial}_s^\alpha \quad (\bar{\partial}_s^\alpha := \partial / \partial \bar{y}_\alpha^s) \\ \text{(c)} \quad d\bar{x}^i &= \partial_j \bar{x}^i dx^j & \text{(d)} \quad \delta \bar{y}_\alpha^i &= \partial_j \bar{x}^i \delta y_\alpha^j. \end{aligned}$$

By a direct calculation we obtain

$$(3.8) \quad \text{(a)} \quad [\delta_j, \delta_k] = R^a_{jk} \partial_a \quad \text{(b)} \quad [\delta_j, \partial_b] = \partial_b N^a_j \partial_a$$

where

$$(3.9) \quad R^a_{jk} = \delta_k N^a_j - \delta_j N^a_k.$$

Thus the horizontal distribution $u \rightarrow H_u E$ is integrable iff $R^a_{jk} = 0$.

Definition 3.2. The tensor algebra spanned by $1, \delta_i, \partial_a, dx^i, \delta y^a$ is called the algebra of d -tensor fields over $\mathcal{F}(E)$.

For convenience we give examples of tensor fields of type $(1, 1), (2, 0)$ and $(0, 2)$: $t^i_j \delta_i \otimes dx^j, t^a_b \partial_a \otimes \delta y^b, t^{ia} \delta_i \otimes \partial_a, t^{ab} \partial_a \otimes \partial_b, t_{aj} \delta y^a \otimes dx^j$.

Remark 3.2. All the coefficients of these tensor products change like the coefficients of a tensor field on M with respect to the Latin indices, the Greek indices being unchanged.

Remark 3.3. The functions R^a_{jk} define the d -tensor field $R = R^a_{jk} dx^j \otimes dx^k \otimes \partial_a$ which is called the *integrability tensor* of the horizontal distribution (cf. [6]).

4. Tensorial structures on $E = \bigoplus_1^k TM$

Let us suppose that there exists on E a nonlinear connection such that (3.1) holds. Then two *supplementary projectors* ν, h and an *almost*

product structure $P = h - \nu$ can be considered. Locally these operators are as follows:

$$(4.1) \quad \begin{array}{ll} (a) & \nu(\delta_i) = 0 \\ (b) & \nu(\partial_a) = \partial_a \\ (c) & h(\delta_i) = \delta_i \\ (d) & h(\partial_a) = 0 \\ (e) & P(\delta_i) = \delta_i \\ (f) & P(\partial_a) = -\partial_a. \end{array}$$

It is easy to check that the following equalities hold

$$(4.2) \quad \overset{\alpha}{J}P = \overset{\alpha}{J}; \quad P\overset{\alpha}{J} = -\overset{\alpha}{J}$$

for every α .

For P we have

Theorem 4.1. *If $P : T_u E \rightarrow T_u E$ ($u \in E$) is an endomorphism satisfying (4.2) then $P^2 = I$ and the eigenspace corresponding to the eigenvalue -1 is a vertical subspace.*

PROOF.

A. P can be expressed locally as follows:

$$P(\partial_i) = P^j_i \partial_j + P^a_i \partial_a; \quad P(\partial_a) = P_a^j \partial_j + P_a^b \partial_b.$$

Then $\overset{\alpha}{J}P = \overset{\alpha}{J}$ yields $P^j_i = \delta^j_i$ and $P_a^j = 0$. Moreover, $P\overset{\alpha}{J} = -\overset{\alpha}{J}$ implies that $P(\partial_a) = -\partial_a$. Hence $P(\partial_i) = \partial_i + P^a_i \partial_a$ and $P(\partial_a) = -\partial_a$. Using these expressions a short calculation shows that $P^2 = I$.

B. Suppose that $PX = -X$ ($X \in T_u E$). From this and $\overset{\alpha}{J}P = \overset{\alpha}{J}$ we obtain $\overset{\alpha}{J}X = \overset{\alpha}{J}PX = \overset{\alpha}{J}(-X) = -\overset{\alpha}{J}X$ and hence $\overset{\alpha}{J}X = 0$. This implies $X^i = 0$ in $X = X^i \partial_i + X^a \partial_a$, i.e. $X \in V_u E$.

To a nonlinear connection we can associate a set of F -structures in K. Yano's sense ([13]). Indeed, if we put

$$(4.3) \quad \begin{array}{l} \overset{\alpha}{F}(\delta_i) = -\partial_i^\alpha \\ \overset{\alpha}{F}(\partial_i^\beta) = \delta^{\alpha\beta} \delta_i \end{array} \quad \text{where } \delta^{\alpha\beta} = \begin{cases} 1, & \alpha = \beta \\ 0, & \alpha \neq \beta \end{cases}$$

we get k tensor fields of type $(1, 1)$ on E which satisfy

$$(4.4) \quad \overset{\alpha}{F}^3 + \overset{\alpha}{F} = 0 \quad (\alpha = \overline{1, k})$$

as it is easy to check.

Another set of *tensorial structures* on E can be defined as follows:

$$(4.5) \quad \overset{\alpha}{Q}(\delta_i) = \partial_i^\alpha, \quad \overset{\alpha}{Q}(\partial_i^\alpha) = \delta_i, \quad \overset{\alpha}{Q}(\partial_i^\beta) = 0 \quad (\alpha \neq \beta).$$

We can easily calculate that

$$(4.6) \quad \overset{\alpha}{Q}^3 - \overset{\alpha}{Q} = 0.$$

In the last part of the next section we study the *integrability* of these structures.

5. d -connections on $E = \bigoplus_1^k TM$

Let us suppose that E is endowed with a nonlinear connection.

Definition 5.1. A linear connection on E is a map $C : TE \rightarrow E$ for which

$$C(x, y, X, \dot{X}) = (x, \dot{X}^a + K_{bi}^a(x)X^i y^b) \quad (a, b = \overline{1, nk})$$

holds.

Remark 5.1. The real functions $K_{bi}^a(x)$ defined on M determine a linear connection D on E ([6]).

Definition 5.2. A linear connection D on E is said to be a distinguished connection (shortly d -connection) if D preserves by parallelism the vertical distribution $u \rightarrow V_u E$ as well as the horizontal distribution $u \rightarrow H_u E$.

Theorem 5.1. A linear connection D on E is a d -connection iff one of the following conditions holds:

$$(5.1) \quad \begin{aligned} & (a) \quad vD_X(hY) = 0, \quad hD_X(vY) = 0 \quad (X, Y \in \mathcal{X}(E)) \\ & (b) \quad Dv = 0, \quad Dh = 0 \\ & (c) \quad DP = 0 \\ & (d) \quad D_X Y = hD_X(hY) + vD_X(vY) \quad (X, Y \in \mathcal{X}(E)) \\ & (e) \quad vD_X(h\omega) = 0, \quad hD_X(v\omega) = 0 \quad (\omega \in \Lambda^1(E), \quad X \in \mathcal{X}(E)) \\ & (f) \quad D_X \omega = hD_X(h\omega) + vD_X(v\omega). \end{aligned}$$

PROOF.

A. Suppose that D is a d -connection. The Definition 5.2. gives $D_X(hY) \in HE$ and $D_X(vY) \in VE$. Hence we directly get the condition (5.1) (a). The conditions $Dv = 0$ and $Dh = 0$ are equivalent to $D_X(vY) = vD_X Y$ and to $D_X(hY) = hD_X Y$ respectively. Since we have an almost product structure $P = h-v$ the condition (c) is equivalent to (b). Moreover, since $D_X Y = hD_X(hY) + vD_X(hY) + hD_X(vY) + vD_X(vY)$ (cf. [6]) and D is a d -connection we get the condition (d). The conditions (e) and (f) are analogous to (a) and (d).

B. A direct calculus shows that any condition in (5.1) is sufficient in order that D should be a d -connection.

The following decomposition holds and is unique for every $X, Y \in \mathcal{X}(E)$:

$$(5.2) \quad D_X Y = D_{hX} Y + D_{vX} Y.$$

Putting

$$(5.3) \quad D_X^h Y = D_{hX} Y, \quad D_X^h \omega = D_{hX} \omega, \quad D_X^h(f) = (hX)f$$

for $X, Y \in \mathcal{X}(E)$, $\omega \in \Lambda^1(E)$, $f \in \Lambda^0(E)$ (cf. [6]) and extending this operator by linearity and Leibniz rule one obtains an *operator of covariant derivation* in the algebra of the d -tensor fields over E called the *h -covariant derivation*. Similarly putting

$$(5.4) \quad D_X^v Y = D_{vX} Y, \quad D_X^v \omega = D_{vX} \omega, \quad D_X^v(f) = (vX)f$$

we obtain the operator of the *v -covariant derivation* in the same algebra.

In local coordinates D^h and D^v , respectively appear as follows:

$$(5.5) \quad \begin{aligned} (a) \quad & D_{\delta_k}^h \delta_j = L_{jk}^i \delta_i; \quad D_{\delta_k}^h \partial_j^\beta = L_{\alpha j k}^{i\beta} \partial_i^\alpha; \quad D_{\delta_k}^h f = \delta_k f \\ (b) \quad & D_{\partial_k^\beta}^v \delta_j = C_{jk}^{i\beta} \delta_i; \quad D_{\partial_k^\beta}^v \partial_j^\gamma = C_{j\alpha k}^{\gamma i\beta} \partial_i^\alpha; \quad D_{\partial_k^\beta}^v f = \partial_k^\beta f. \end{aligned}$$

So we obtain a set of functions defined locally on E

$$(5.6) \quad D\Gamma = (L_{jk}^i(x, y), L_{\alpha j k}^{i\beta}(x, y), C_{jk}^{i\beta}(x, y), C_{j\alpha k}^{\gamma i\beta}(x, y))$$

which gives a d -connection D .

Let $x^i = x^i(\bar{x})$ and $\bar{x}^i = \bar{x}^i(x)$ respectively be a transformation of the local coordinates on a neighbourhood of M . Then the above coefficients change as follows:

$$(5.7) \quad \begin{aligned} \bar{L}_{jk}^i &= \partial_s \bar{x}^i \bar{\partial}_j x^m \bar{\partial}_k x^n L_{mn}^s - \partial_m \partial_n \bar{x}^i \bar{\partial}_j x^m \bar{\partial}_k x^n; \\ \bar{L}_{\alpha j k}^{i\beta} &= \partial_s \bar{x}^i \bar{\partial}_j x^m \bar{\partial}_k x^n L_{\alpha mn}^{s\beta} - \partial_m \partial_n \bar{x}^i \bar{\partial}_j x^m \bar{\partial}_k x^n; \\ \bar{C}_{jk}^{i\beta} &= \partial_s \bar{x}^i \bar{\partial}_j x^m \bar{\partial}_k x^n C_{mn}^{s\beta}; \\ \bar{C}_{\alpha j k}^{i\gamma\beta} &= \partial_s \bar{x}^i \bar{\partial}_j x^m \bar{\partial}_k x^n C_{\alpha mn}^{s\gamma\beta}. \end{aligned}$$

Theorem 5.2. *The formulae (5.7) characterize the coefficients of a d -connection. If a set of functions $D\Gamma$ satisfying (5.7) is given on E then by (5.5) (a), (b) and (5.7) we obtain h - and v -covariant derivatives, and by (5.2) a d -connection on E .*

We give the local form of the h - and v -covariant derivatives of some tensor fields:

h -covariant derivative:

$$t_{a|k}^i = \delta_k t_a^i + L_{sk}^i t_a^s - L_{ak}^b t_b^i.$$

v -covariant derivative:

$$t_b^a|_c = \partial_c t_b^a + C_{dc}^a t_b^d - C_{bc}^d t_d^a.$$

It is obvious that the vector field $C = y^a \partial_a$ is globally defined on E .

Definition 5.3. A d -connection D on E is said to be of Cartan type if

$$(a) \quad D_X^h C = 0, \quad (b) \quad D_X^v C = vX$$

hold for every $X \in \mathcal{X}(E)$ and

$$(5.8) \quad (c) \quad D^a_k = y^b L_{bk}^a - N^a_k = 0.$$

D^a_k is called the *deflection tensor* of D .

Theorem 5.3. A d -connection D on E is of Cartan type iff

$$(5.9) \quad D^a_k = 0 \text{ and } y^a C_{ab}^c = 0.$$

Indeed, the conditions in (5.9) are equivalent to (5.8) (a), (b), (c).

The *torsion* of a d -connection D on E is defined as usual:

$$\mathbf{T}(X, Y) := D_X Y - D_Y X - [X, Y]; \quad (X, Y \in \mathcal{X}(E)).$$

Decomposition into vertical and horizontal parts leads to the following *five* d -tensor fields which will be called the *torsion tensors* of D :

$$(5.10) \quad \begin{aligned} T(X, Y) &= h\mathbf{T}(hX, hY) = D_X^h(hY) - D_Y^h(hX) - h[hX, hY]; \\ R(X, Y) &= v\mathbf{T}(hX, hY) = -v[hX, hY]; \\ C(X, Y) &= h\mathbf{T}(hX, vY) = -D_Y^v(hX) - h[hX, vY]; \\ P(X, Y) &= v\mathbf{T}(hX, vY) = D_X^h(vY) - v[hX, vY]; \\ S(X, Y) &= v\mathbf{T}(vX, vY) = D_X^v(vY) - D_Y^v(vX) - v[vX, vY]. \end{aligned}$$

($X, Y \in \mathcal{X}(E)$).

In local coordinates we get

$$\begin{aligned} T(\delta_j, \delta_k) &= T_{kj}^i \delta_i; \quad R(\delta_k, \delta_j) = R_{jk}^a \partial_a; \quad C(\delta_k, \partial_b) = C_{kb}^i \delta_i \\ P(\delta_k, \partial_b) &= P_{bk}^a \partial_a; \quad S(\partial_b, \partial_c) = C_{bc}^a \partial_a \end{aligned}$$

and the *torsion tensor fields* of the d -connection D are

$$(5.11) \quad \begin{aligned} (a) \quad T_{kj}^i &= L_{kj}^i - L_{jk}^i & (b) \quad R_{jk}^a &= \delta_j N_k^a - \delta_k N_j^a \\ (c) \quad C_{kb}^i & & (d) \quad P_{bk}^a &= \partial_b N_k^a - L_{bk}^a & (e) \quad S_{bc}^a &= C_{bc}^a - C_{cb}^a. \end{aligned}$$

In the usual way we get six *curvature tensors*. These are the following ones in local form:

$$(5.12) \quad \begin{aligned} R(\delta_k, \delta_j) \delta_l &= R_{ljk}^i \delta_i, & R(\delta_k, \delta_j) \partial_b &= \tilde{R}_b^a{}_{jk} \partial_a, \\ R(\partial_c, \delta_k) \delta_j &= P_{jkc}^i \delta_i, & R(\partial_c, \delta_k) \partial_b &= \tilde{P}_b^a{}_{kc} \partial_a, \\ R(\partial_c, \partial_b) \delta_j &= S_{jbc}^i \delta_i, & R(\partial_d, \partial_c) \partial_b &= \tilde{S}_b^a{}_{cd} \partial_a, \end{aligned}$$

Next we define a *particular case* of d -connection.

Definition 5.4. A linear d -connection $D\Gamma = (L_{jk}^i, L_{\alpha kj}^{i\beta}, C_{jk}^{i\beta}, C_{\alpha jk}^{i\gamma\beta})$ on E is normal if $D\bar{F}^\alpha = 0$, i.e. the F -structures $\bar{F}^\alpha$ are absolute parallel by D for every α .

Since $D_X(\bar{F}^\alpha Y) = (D_X \bar{F}^\alpha)Y + \bar{F}^\alpha(D_X Y)$ the condition $D\bar{F}^\alpha = 0$ is equivalent to

$$(5.13) \quad D_X(\bar{F}^\alpha Y) = \bar{F}^\alpha(D_X Y).$$

From this we obtain in local coordinates for h -covariant derivatives:

$$(5.14) \quad \begin{aligned} (a) \quad & D_{\delta_k}^h \bar{F}^\alpha(\delta_j) = \bar{F}^\alpha(D_{\delta_k}^h \delta_j), \\ \text{i.e.} \quad (b) \quad & L_{\beta jk}^{i\alpha} \partial_i^\beta = L_{jk}^i \partial_i^\alpha. \end{aligned}$$

Moreover, we have

$$(5.15) \quad \begin{aligned} (a) \quad & D_{\delta_k}^h \bar{F}^\alpha(\partial_j^\alpha) = \bar{F}^\alpha(D_{\delta_k}^h \partial_j^\alpha), \\ \text{i.e.} \quad (b) \quad & \delta_\beta^\alpha L_{jk}^s \delta_s = L_{j\alpha k}^{\beta s} \delta_s. \end{aligned}$$

So we obtain

$$(5.16) \quad L_{j\alpha i}^{\beta s} = \delta_\alpha^\beta L_{ji}^s.$$

If $\alpha \neq \beta$ then

$$(5.17) \quad D_{\delta_k}^h \bar{F}^\alpha(\partial_j^\beta) = \bar{F}^\alpha(D_{\delta_k}^h \partial_j^\beta) = 0.$$

We can carry out similar calculations for v -covariant derivatives:

$$(5.18) \quad \begin{aligned} (a) \quad & D_{\partial_i^\beta}^v \bar{F}^\alpha(\delta_j) = \bar{F}^\alpha(D_{\partial_i^\beta}^v \delta_j), \\ \text{i.e.} \quad (b) \quad & C_{j\gamma i}^{\alpha l\beta} \partial_l^\gamma = C_{ji}^{s\beta} \partial_s^\alpha. \end{aligned}$$

The next step is the following

$$(5.19) \quad \begin{aligned} (a) \quad & D_{\partial_i^\beta}^v \bar{F}^\alpha(\partial_j^\gamma) = \bar{F}^\alpha(D_{\partial_i^\beta}^v \partial_j^\gamma) \quad (\alpha = \gamma), \\ \text{i.e.} \quad (b) \quad & \delta_\epsilon^\gamma C_{ji}^{s\beta} \delta_s = C_{j\epsilon i}^{\gamma s\beta} \delta_s. \end{aligned}$$

Finally we get

$$(5.20) \quad C_{j\epsilon i}^{\gamma s\beta} = \delta_\epsilon^\gamma C_{ji}^{s\beta}.$$

Hence we have

Theorem 5.4. A linear d -connection $D\Gamma$ is normal iff

$$(5.21) \quad \begin{array}{ll} \text{(a)} & L_{\alpha ji}^{s\beta} = \delta_{\alpha}^{\beta} L_{ji}^s \\ \text{and} & \text{(b)} & C_{j\epsilon i}^{\gamma s\beta} = \delta_{\epsilon}^{\gamma} C_{ji}^{s\beta} \end{array}$$

hold.

Thus a normal d -connection is completely determined by $(L_{jk}^i, C_{jk}^{i\beta})$.

Its torsions are as follows:

$$(5.22) \quad \begin{array}{lll} \text{(a)} & T_{kj}^i = L_{kj}^i - L_{jk}^i & \text{(b)} \quad T_{\alpha kj}^i = R_{\alpha kj}^i \quad \text{(c)} \quad C_{jk}^{i\beta} \\ \text{(d)} & P_{jik}^{\beta\alpha} = \partial_j^{\beta} N_{\alpha k}^i - \delta_{\alpha}^{\beta} L_{jk}^i & \text{(e)} \quad S_{j\gamma i}^{\beta k\alpha} = \delta_{\gamma}^{\beta} C_{ji}^{k\alpha} - \delta_{\gamma}^{\alpha} C_{ij}^{k\beta}. \end{array}$$

The number of curvatures reduces to three instead of six.

In the previous section we have defined a set of F -structures $\overset{\alpha}{F}$.

Remark 5.2. Because of

$$(5.23) \quad \begin{array}{ll} \text{(a)} & \overset{\alpha}{F}\overset{\beta}{F}(\delta_i)_{(\alpha \neq \beta)} = -\delta^{\alpha\beta} \delta_i = 0, \quad \text{(b)} \quad \overset{\beta}{F}\overset{\alpha}{F}(\delta_i)_{(\alpha \neq \beta)} = -\delta^{\alpha\beta} \delta_i = 0, \\ \text{(c)} & \overset{\alpha}{F}\overset{\beta}{F}(\partial_i^{\gamma}) = -\delta^{\beta\gamma} \partial_i^{\alpha}, \quad \text{(d)} \quad \overset{\beta}{F}\overset{\alpha}{F}(\partial_i^{\gamma}) = -\delta^{\alpha\gamma} \partial_i^{\beta} \end{array}$$

we get

$$(5.24) \quad \overset{\alpha}{F}\overset{\beta}{F} \neq \overset{\beta}{F}\overset{\alpha}{F} \quad \text{if} \quad \alpha \neq \beta.$$

On the other hand we have

$$(5.25) \quad \overset{\alpha}{F}^4 = -\overset{\alpha}{F}^2.$$

(5.25) shows that the operators $P_1 = -\overset{\alpha}{F}^2$, $P_2 = I + \overset{\alpha}{F}^2$ are two supplementary projectors, and taking their kernels one obtains two distributions $\mathcal{D}_1$ and $\mathcal{D}_2$ which are spanned locally by $\{\partial_i^{\beta}\}$ ($\beta \neq \alpha$) and $(\delta_i, \partial_i^{\alpha})$, respectively.

We define in the sense of V. DUC ([2]) that $\overset{\alpha}{F}$ is integrable if $\mathcal{D}_1$ and $\mathcal{D}_2$ are involutive. He treated a general F structure and proved that it is integrable iff the *Nijenhuis tensor* of its square vanishes.

Next we prove

Proposition 5.1. *The Nijenhuis tensor of $\overset{\alpha}{F}^2$ is equal to zero iff*

$$(5.26) \quad (a) \quad R_{\beta ij}^k = 0 \quad \text{and} \quad (b) \quad \partial_j^\alpha (N_{\beta i}^s) = 0$$

for any $\beta \neq \alpha$.

PROOF. The Nijenhuis tensor of $\overset{\alpha}{F}^2$ is

$$(5.27) \quad N_{\overset{\alpha}{F}^2}(X, Y) = \left[\overset{\alpha}{F}^2 X, \overset{\alpha}{F}^2 Y \right] - \overset{\alpha}{F}^2 \left[\overset{\alpha}{F}^2 X, Y \right] - \overset{\alpha}{F}^2 \left[X, \overset{\alpha}{F}^2 Y \right] + \overset{\alpha}{F}^4 [X, Y] \quad (X, Y) \in \mathcal{X}(E).$$

Hence by using (5.10), (5.24) and (5.25) we get for the adapted basis $(\delta_i, \partial_i^\alpha)$ the following equalities:

$$\begin{aligned} (a) \quad N_{\overset{\alpha}{F}^2}(\delta_i, \delta_j) &= \left[\overset{\alpha}{F}^2 \delta_i, \overset{\alpha}{F}^2 \delta_j \right] - \overset{\alpha}{F}^2 \left[\overset{\alpha}{F}^2 \delta_i, \delta_j \right] - \overset{\alpha}{F}^2 \left[\delta_i, \overset{\alpha}{F}^2 \delta_j \right] + \\ &\quad + \overset{\alpha}{F}^4 [\delta_i, \delta_j] = [\delta_i, \delta_j] - \overset{\alpha}{F}^2 \left[-\overset{\alpha}{F} \partial_i^\alpha, \delta_j \right] - \overset{\alpha}{F}^2 \left[\delta_i, -\overset{\alpha}{F} \partial_j^\alpha \right] - \\ &\quad - \overset{\alpha}{F}^2 [\delta_i, \delta_j] = R_{\beta ij}^k \partial_k^\beta + \overset{\alpha}{F}^2 [\delta_i, \delta_j] + \overset{\alpha}{F}^2 [\delta_i, \delta_j] - \overset{\alpha}{F}^2 [\delta_i, \delta_j] = \\ &= R_{\beta ij}^k \partial_k^\beta + \overset{\alpha}{F}^2 (R_{\beta ij}^k \partial_k^\beta) = R_{\beta ij}^k \partial_k^\beta + R_{\beta ij}^k \left(\overset{\alpha}{F}^2 \partial_k^\beta \right) = \\ &= R_{\beta ij}^k \partial_k^\beta + R_{\beta ij}^k \overset{\alpha}{F} (\delta^{\alpha\beta} \delta_k) = R_{\beta ij}^k \partial_k^\beta - R_{\alpha ij}^k \partial_k^\alpha = R_{\beta ij}^k \partial_k^\beta, \\ &\quad \text{(not summing over } \alpha, \beta \neq \alpha), \\ (b) \quad N_{\overset{\alpha}{F}^2}(\delta_i, \partial_j^\alpha) &= [\delta_i, \partial_j^\alpha] - \overset{\alpha}{F}^2 \left[\delta_i, \overset{\alpha}{F} (\delta^{\alpha\beta} \delta_j) \right] - \\ &\quad - \overset{\alpha}{F}^2 \left[\overset{\alpha}{F} (-\partial_i^\alpha), \partial_j^\alpha \right] - \overset{\alpha}{F}^2 [\delta_j, \partial_j^\alpha] = [\delta_i, \partial_j^\alpha] + \overset{\alpha}{F}^2 [\delta_i, \partial_j^\alpha] - \\ &\quad - \overset{\alpha}{F}^2 [-\delta^{\alpha\beta} \delta_i, \partial_j^\alpha] - \overset{\alpha}{F}^2 [\delta_i, \partial_j^\alpha] = \partial_j^\alpha N_{\beta i}^s \partial_s^\beta + \overset{\alpha}{F}^2 [\delta_i, \partial_j^\alpha] = \\ &= (\partial_j^\alpha N_{\beta i}^s) \partial_s^\beta + (\partial_j^\alpha N_{\beta i}^s) \overset{\alpha}{F} (\delta^{\alpha\beta} \delta_s) = (\partial_j^\alpha N_{\beta i}^s) \partial_s^\beta - (\partial_j^\alpha N_{\alpha i}^s) \partial_s^\alpha = \\ &= (\partial_j^\alpha N_{\beta i}^s) \partial_s^\beta \quad (\beta \neq \alpha, \text{ not summing over } \alpha). \end{aligned}$$

If $\beta \neq \alpha$ then we have

$$(c) \quad N_{\overset{\alpha}{F}^2}(\delta_i, \partial_j^\beta) = [-\delta_i, 0] + \overset{\alpha}{F}^2 [\delta_i, \partial_j^\beta] - \overset{\alpha}{F}^2 [\delta_i, 0] - \overset{\alpha}{F}^2 [\delta_i, \partial_j^\beta] = 0.$$

Moreover, for $\beta \neq \alpha$ and $\gamma \neq \alpha$ respectively we get

$$(5.28) \quad (d) \quad N_{\tilde{F}^2}^\alpha(\partial_i^\alpha, \partial_j^\beta) = [-\partial_i^\alpha, 0] + \tilde{F}^2[\partial_i^\alpha, \partial_j^\beta] - \tilde{F}^2[-\partial_i^\alpha, 0] - \\ - \tilde{F}^2[\partial_i^\alpha, \partial_j^\beta] = 0, \\ (e) \quad N_{\tilde{F}^2}^\alpha(\partial_i^\beta, \partial_j^\gamma) = 0.$$

By linearity of the Nijenhuis tensor these equalities establish our assertion.

Remark 5.2. The condition (5.26) (b) shows that the functions $N_{\alpha i}^s$ do not depend on y_β^j if $\beta \neq \alpha$.

Let us consider the formulae (4.5) and (4.6). We define *three supplementary projectors*. $P_1 = I - \tilde{Q}^2$, $P_2 = \frac{1}{2} \left(\tilde{Q}^2 + \tilde{Q} \right)$ and

$P_3 = \frac{1}{2} \left(\tilde{Q}^2 - \tilde{Q} \right)$. So we have three *distributions* $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$. (V.

Duc ([2]) treated a general structure K with $K^3 = K$.) In his sense $\tilde{Q}^\alpha$ is integrable if the distributions $\mathcal{D}_i + \mathcal{D}_j$ ($i, j = 1, 2, 3$) are involutive. By this theorem $\tilde{Q}^\alpha$ is integrable iff $N_{\tilde{Q}^\alpha} = 0$.

Now we prove

Proposition 5.2. *The Nijenhuis tensor of $\tilde{Q}^\alpha$ is equal to zero iff*

$$(5.29) \quad (a) \quad \partial_i^\alpha N_{\alpha j}^s = \partial_j^\alpha N_{\alpha i}^s \quad (b) \quad R_{\alpha ij}^k = 0 \\ (c) \quad \partial_i^\alpha N_{\beta j}^s = 0 \quad (\beta \neq \alpha) \quad (d) \quad \partial_j^\beta N_{\alpha i}^s = 0 \quad (\beta \neq \alpha).$$

PROOF. By using the relations (4.5) and (5.27) for $N_{\tilde{Q}^\alpha}$ it will be sufficient to calculate that

$$(a) \quad N_{\tilde{Q}^\alpha}(\delta_i, \delta_j) = \left[\tilde{Q}^\alpha \delta_i, \tilde{Q}^\alpha \delta_j \right] - \tilde{Q}^\alpha \left[\tilde{Q}^\alpha \delta_i, \delta_j \right] - \tilde{Q}^\alpha \left[\delta_i, \tilde{Q}^\alpha \delta_j \right] + \\ + \tilde{Q}^2[\delta_i, \delta_j] = [\partial_i^\alpha, \partial_j^\beta] - \tilde{Q}^\alpha[\partial_i^\alpha, \delta_j] - \tilde{Q}^\alpha[\delta_i, \partial_j^\alpha] + \\ + \tilde{Q}^2(R_{\gamma ij}^k \partial_k^\gamma) = \tilde{Q}^\alpha((\partial_i^\alpha N_{\beta j}^s) \partial_s^\beta) - \tilde{Q}^\alpha((\partial_j^\alpha N_{\beta i}^s) \partial_s^\beta) + \\ + R_{\gamma ij}^k \tilde{Q}^\alpha(\delta^{\alpha\gamma} \delta_k) = (\partial_i^\alpha N_{\alpha j}^s - \partial_j^\alpha N_{\alpha i}^s) \delta_s + R_{\alpha ij}^k \partial_k^\alpha, \\ (b) \quad N_{\tilde{Q}^\alpha}(\delta_i, \partial_j^\alpha) = \left[\tilde{Q}^\alpha \delta_i, \tilde{Q}^\alpha \partial_j^\alpha \right] - \tilde{Q}^\alpha \left[\tilde{Q}^\alpha \delta_i, \partial_j^\alpha \right] - \tilde{Q}^\alpha \left[\delta_i, \tilde{Q}^\alpha \partial_j^\alpha \right] +$$

$$\begin{aligned}
& + \tilde{Q}^2[\delta_i, \partial_j^\alpha] = [\partial_i^\alpha, \delta_j] - \tilde{Q}[\partial_i^\alpha, \partial_j^\alpha] - \tilde{Q}[\delta_i, \delta_j] + \\
& + \tilde{Q}^2((\partial_j^\alpha N_{\beta i}^s) \partial_s^\beta) = -(\partial_i^\alpha N_{\beta s}^s) \partial_s^\beta - R_{\gamma ij}^k \left(\tilde{Q} \partial_k^\gamma \right) + \\
& + (\partial_j^\alpha N_{\beta i}^s) \tilde{Q}(\delta^{\alpha\beta} \delta_s) = (\partial_j^\alpha N_{\alpha i}^s) \partial_s^\alpha - (\partial_i^\alpha N_{\beta j}^s) \partial_s^\beta - R_{\alpha ij}^k \delta_k = \\
(5.30) \quad & = -\partial_i^\alpha N_{\beta j}^s \partial_s^\beta - R_{\alpha ij}^k \delta_k \text{ (not summing over } \alpha; \beta \neq \alpha);
\end{aligned}$$

for $\beta \neq \alpha$ we have

$$\begin{aligned}
(c) \quad N_{\tilde{Q}}^\alpha(\delta_i, \partial_j^\beta) &= \left[\tilde{Q} \delta_i, \tilde{Q} \partial_j^\beta \right] - \tilde{Q} \left[\tilde{Q} \delta_i, \partial_j^\beta \right] - \tilde{Q} \left[\delta_i, \tilde{Q} \partial_j^\beta \right] + \\
& + \tilde{Q}^2 \left[\delta_i, \partial_j^\beta \right] = [\partial_i^\alpha, 0] - \tilde{Q} \left[\partial_i^\alpha, \partial_j^\beta \right] - \tilde{Q}[\delta_i, 0] + \\
& + \tilde{Q}^2((\partial_j^\beta N_{\gamma i}^s) \partial_s^\gamma) = (\partial_j^\beta N_{\gamma i}^s) \tilde{Q}(\delta^{\alpha\gamma} \delta_s) = (\partial_j^\beta N_{\alpha i}^s) \partial_s^\alpha \\
& \text{(not summing over } \alpha),
\end{aligned}$$

and

$$(d) \quad N_{\tilde{Q}}^\alpha(\partial_i^\alpha, \partial_j^\beta) = 0, \quad (e) \quad N_{\tilde{Q}}^\alpha(\partial_j^\beta, \partial_k^\gamma) = 0 \quad (\gamma \neq \alpha).$$

These establish our assertion. We can summarize our results in

Theorem 5.5. Any $\tilde{F}^\alpha$ and any $\tilde{Q}$ respectively is integrable iff the conditions (5.26) (a), (b) and (5.29) (a), (b), (c), (d) respectively hold.

Remark 5.2. Integrability of all $\tilde{Q}$ implies integrability of all $\tilde{F}^\alpha$. The converse is not true.

6. Geometry of Lagrangians on $E = \bigoplus_1^k TM$

Let M be endowed with a nonlinear connection.

Definition 6.1. A function $\mathcal{L} : \bigoplus_1^k TM \setminus \{0\} \rightarrow R$ is said to be a regular Lagrangian on E if the matrix with the elements

$$(6.1) \quad g_{ab} = \partial_a \partial_b \mathcal{L}(x^i, y^c)$$

is of rank nk .

Definition 6.2. A k -Lagrange space L_k^n is a pair $(M, \mathcal{L}(x^i, y^a))$ where M is an n -dim. manifold and $\mathcal{L}(x^i, y^a)$ is a regular Lagrangian defined over $E = \bigoplus_1^k TM$.

Proposition 6.1. The set of functions $g_{ab}(x, y)$ and $g_{ij}(x, y)$ defines v - and h -Riemannian structures in the vertical and horizontal bundles as follows:

$$(6.2) \quad \begin{aligned} (a) \quad & g : u \rightarrow g_u : VE \times VE \rightarrow R \quad (u \in E) \\ & g_u(X, Y) = X^a Y^b g_{ab}(x, y) \quad (X, Y \in \mathcal{X}(VE)) \\ (b) \quad & g : u \rightarrow g_u : HE \times HE \rightarrow R \quad (u \in E) \end{aligned}$$

$$g_u(X, Y) = g(X^i \delta_i, Y^j \delta_j) = X^i Y^j g(\delta_i, \delta_j) = X^i Y^j g_{ij}(x, y);$$

$$\det \|g_{ij}\| \neq 0, \quad (X, Y \in \mathcal{X}(HE)).$$

PROOF. Under the conditions $\text{rank} \|g_{ab}\| = nk$ and $\text{rank} \|g_{ij}\| = n$, for each $u \in E$, g_u is a nondegenerate bilinear form on $VE \times VE$ and $HE \times HE$, respectively. Moreover, $g(X, Y) = g(Y, X)$ for all $X, Y \in \mathcal{X}(VE)$ and $\mathcal{X}(HE)$, respectively. These prove the Proposition.

We shall denote by g^{cd} the *inverse* of the matrix g_{ab} , i.e. the following relations hold:

$$(6.3) \quad g_{ab} g^{bc} = \delta_a^c, \quad g^{cd} g_{da} = \delta_a^c$$

where δ_a^c are the components of the Kronecker tensor on E .

Let us suppose that we have h - and v -Riemannian structures, i.e. $g_{ij}(x, y)$ and $g_{ab}(x, y)$. Then the following *metric* can be considered on E

$$(6.4) \quad G = g_{ij}(x, y) dx^i \otimes dx^j + g_{ab}(x, y) \delta y^a \otimes \delta y^b.$$

Definition 6.3. A d -connection $LD = (L_{jk}^i, L_{i\beta k}^{\alpha j}, C_{jk}^{i\alpha}, C_{k\alpha j}^{\gamma i\beta})$ is called metrical with respect to G if

$$(6.5) \quad g_{ij|k} = 0, \quad g_{ij}|_k^\alpha = 0, \quad g_{ab|k} = 0, \quad g_{ab}|_k^\alpha = 0,$$

where the short and long bars mean h - and v -covariant derivatives.

Theorem 6.1. *The following d -connection is metrical and its torsion tensors T and S vanish:*

$$\begin{aligned}
 (a) \quad & L_{jk}^i = \frac{1}{2} g^{il} (\delta_j g_{kl} + \delta_k g_{jl} - \delta_l g_{jk}) \\
 (b) \quad & L_{i\beta k}^{\alpha j} = \partial_i^\alpha (N_{\beta k}^j) + \frac{1}{2} g_{\beta\gamma}^{js} (\delta_k g_{is}^{\alpha\gamma} - \partial_i^\alpha (N_{\epsilon k}^r) g_{rs}^{\epsilon\gamma} - \partial_s^\gamma (N_{\epsilon k}^r) g_{ri}^{\epsilon\alpha}) \\
 (c) \quad & C_{jk}^{i\beta} = \frac{1}{2} g^{is} \partial_k^\beta (g_{js}) \\
 (d) \quad & C_{k\alpha j}^{\gamma i\beta} = \frac{1}{2} g_{\alpha\epsilon}^{is} (\partial_k^\gamma g_{js}^{\beta\epsilon} + \partial_j^\beta g_{ks}^{\gamma\epsilon} - \partial_s^\epsilon g_{kj}^{\gamma\beta}).
 \end{aligned}
 \tag{6.6}$$

PROOF. We can easily see that L_{jk}^i and $C_{k\alpha j}^{\gamma i\beta}$ are *symmetric* in j, k and $(\gamma), (\beta)$, respectively. Moreover, taking into account the relations (5.11) (a) and (e), we get $T = S = 0$. The equalities in (6.5) can be checked by a direct calculus.

Remark 6.1. If M is endowed with a metric $\tilde{g}_{ij}(x)$ then we can define its horizontal lift $g_u(x, y) = X^i Y^j g_{ij}(x)$. In this case the coefficients of LD are simpler, since the $\tilde{g}_{ij}$ depend on x only not on y . We want to investigate this case. We obtain $C_{jk}^{i\beta} = 0$. On the other hand

$$C_{k\alpha j}^{\gamma i\beta} = \frac{1}{2} g_{\alpha\epsilon}^{is} \frac{\partial^3 \mathcal{L}}{\partial y_\beta^j \partial y_\gamma^k \partial y_\epsilon^s}; \quad g_{si}^{\epsilon\alpha} C_{k\alpha j}^{\gamma i\beta} := \tilde{C}_{ksj}^{\gamma\epsilon\beta} = \frac{1}{2} \frac{\partial^3 \mathcal{L}}{\partial y_\beta^j \partial y_\gamma^k \partial y_\epsilon^s}.$$

It follows that $\tilde{C}_{ksj}^{\gamma\epsilon\beta}$ is symmetric in the pair of indices $(\epsilon), (\gamma), (\beta)$. Moreover we have $L_{jk}^i = \frac{1}{2} g^{il} (\partial_j g_{kl} + \partial_k g_{jl} - \partial_l g_{jk})$.

Remark 6.2. The torsion tensors T, S, C of LD considered in Remark 6.1. vanish.

Remark 6.3. Generally $R \neq 0$, and from (5.11) (a) and (6.6) (b) we get

$$P_{j\alpha k}^{\beta i} = \partial_j^\beta (N_{\alpha k}^i) - L_{j\alpha k}^{\beta i} = -\frac{1}{2} g_{\alpha\gamma}^{is} (\delta_k g_{js}^{\beta\gamma} - \partial_j^\beta (N_{\epsilon k}^r) g_{rs}^{\epsilon\gamma} - \partial_s^\gamma (N_{\epsilon k}^r) g_{rj}^{\epsilon\beta}).$$

This implies

$$g_{si}^{\gamma\alpha} P_{j\alpha k}^{\beta i} = -\frac{1}{2} (\delta_k g_{js}^{\beta\gamma} - \partial_j^\beta (N_{\epsilon k}^r) g_{rs}^{\epsilon\gamma} - \partial_s^\gamma (N_{\epsilon k}^r) g_{rj}^{\epsilon\beta}).$$

Denote $\tilde{P}_{jsk}^{\beta\gamma} = g_{si}^{\gamma\alpha} P_{j\alpha k}^{\beta i}$ and we see that $\tilde{P}$ is symmetric in (β) and (γ) .

Let D be a d -connection on $E = \bigoplus_1^k TM$. C_t a curve of M ,
 $\tilde{C} : C_t \rightarrow \bigoplus_1^k TM$ a section of E over C_t and $X(x^i, y_\alpha^i)$ a k -Lagrangian
 vector field on $E = \bigoplus_1^k TM$.

Definition 6.3. The covariant derivative of the vector field X on $\tilde{C}$ with respect to D is

$$(6.7) \quad \frac{DX}{dt^\alpha} := (D_{\delta_i}^h X) \frac{dx^i}{dt^\alpha} + (D_{\partial_k^\beta}^v X) \frac{\delta y_\beta^k}{dt^\alpha}.$$

Definition 6.4. A k -Lagrangian vector field X in $\mathcal{X}(E)$ is called parallel on the section $\tilde{C} : C_t \rightarrow E = \bigoplus_1^k TM$ if

$$(6.8) \quad \frac{DX}{dt^\alpha} = 0 \quad (\alpha = \overline{1, k})$$

(cf. [7]).

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MAGDALEN SZ. KIRKOVITS
UNIV. OF FORESTRY AND WOOD SCIENCES
DEPARTMENT OF MATHEMATICS
9400—SOPRON
HUNGARY

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