

Estimate of the rate of mean convergence of Hermite interpolating processes

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The mean convergence of interpolating processes was first investigated in the famous work [1] of P. ERDŐS and P. TURÁN in 1937 on a finite interval and this was generalized for an infinite interval in 1961 by J. BALÁZS and P. TURÁN [2]. In [3] JOÓ and SZABADOS proved an L^1 -convergence theorem for interpolating processes based on Laguerre nodes, with estimate for the rate of convergence and this was generalized and strengthened in [4] by I. JOÓ. Probably, the results given in [4] are not refinable.

The aim of the present paper is to generalize the Theorem of JOÓ on the Hermite interpolating process based on Laguerre nodes in two directions: first we give a different form for the rate of convergence, and secondly, our method works for more general nodes than the Laguerrian one.

Denote by $L_n^{(\alpha)}(x) := x^{-\alpha} e^x \frac{1}{n!} [e^{-x} x^{n+\alpha}]^{(n)}$ the Laguerre polynomials ($\alpha > -1$) and by $x_1, \dots, x_n$ the zeros of $L_n^{(\alpha)}(x)$ (in fact $x_k = x(k, n, \alpha)$; but we simplify the notation). Define by $\ell_k(x) := L_n^{(\alpha)}(x) / L_n^{(\alpha)'}(x_k)(x - x_k)$ the fundamental polynomials of the Lagrange interpolation based on the roots of $L_n^{(\alpha)}(x)$ and put

$$F_n(f, x) := \sum_{k=1}^n \left[f(x_k) \frac{x_k(x_k - \alpha) + x(\alpha + 1 - x_k)}{x_k} + f'(x_k)(x - x_k) \right] \ell_k^2(x).$$

Introduce the space $C(\lambda) := \{f \in C[0, \infty) : \lim_{x \rightarrow \infty} f(x)e^{-\lambda x} = 0\}$, and write $\omega(f, \delta) := \omega_\infty(f, \delta) := \sup_{\substack{x, y \in \mathbf{R} \\ |x-y| \leq \delta}} |f(x) - f(y)|$ for the modulus of continuity of f .

We shall prove the following

Theorem. Suppose $\alpha > -1$, $0 < \lambda < \mu < 1$; $f, f' \in C(\lambda)$. Then

$$(1) \quad \int_0^\infty x^\alpha e^{-x} |f(x) - F_n(f, x)| dx \leq \\ \leq C \left\{ \omega \left(F^*, \frac{a}{n} \right) + \left(\frac{\lambda}{\mu} \right)^b + 2^{\frac{n}{r}} r n^\alpha e^{-6n} \right\}$$

where $r \geq 1$ (may depend on n), $a = \left(\frac{6n}{1-\mu} \right)^{1/r}$, $b = n(1 - r^{-1})$, $F^*(x) = f'(|x|^r) e^{-\lambda|x|^r}$ and the constant C doesn't depend on n, r, f .

For the proof we need some lemmas.

Lemma 1. Given $0 < \lambda < \mu < 1$, $r \geq 1$ (r may depend on n), $a := \left(\frac{6n}{1-\mu} \right)^{1/r}$, $b = n(1 - \frac{1}{r})$; $f, f' \in C(\lambda)$, there exists a polynomial $p(x) \in \Pi_{n+1}$ such that

$$(a) \quad |f'(x) - p'(x)| = \mathcal{O}(e^{\mu x}) \left\{ \omega \left(F^*, \frac{a}{n} \right) + \left(\frac{\lambda}{\mu} \right)^b \right\} \quad (0 \leq x \leq a^r),$$

$$(b) \quad \int_0^\infty x^\alpha e^{-x} |f'(x) - p'(x)| dx = \\ = \mathcal{O} \left\{ \omega \left(F^*, \frac{a}{n} \right) + \left(\frac{\lambda}{\mu} \right)^b + 2^{\frac{n}{r}} a^{r\alpha} e^{-(1-\lambda)a^r} \right\}$$

PROOF.

(a) We have to modify the definition of $p(x)$ in [3], (6) so as to obtain

$$p'(x) = q(\sqrt[r]{x}) \sum_{k=0}^b \frac{(\lambda x)^k}{k!}.$$

(b) According to [3], using the estimate

$$\int_s^\infty e^{-t} t^\beta dt = e^{-s} s^\beta (1 + o(s^{-1})), \quad \beta \in \mathbf{R}, \quad s \rightarrow +\infty,$$

we obtain

$$\int_{a^r}^\infty x^\alpha e^{-x} |f'(x) - p'(x)| dx = \mathcal{O} \left(\left(\frac{2}{a^r} \right)^{\frac{n}{r}} \right) \int_{a^r}^\infty x^{\frac{n}{r} + \alpha} e^{-(1-\lambda)x} dx = \\ = \mathcal{O} \left(2^{\frac{n}{r}} a^{r\alpha} e^{-(1-\lambda)a^r} \right).$$

Lemma 2. *If the conditions of Lemma 1 are fulfilled, then*

$$\int_0^{\infty} x^{\alpha} e^{-x} |f(x) - p(x)| dx = \mathcal{O} \left\{ \omega \left(F^*, \frac{a}{n} \right) + \left(\frac{\lambda}{\mu} \right)^b + 2^{\frac{n}{r}} r a^{r\alpha} e^{-(1-\mu)a^r} \right\}.$$

PROOF. We may suppose that $f(0) = p(0)$ and so

$$f(x) - p(x) = \int_0^x (f'(t) - p'(t)) dt.$$

In case $0 \leq x \leq a^r$, using Lemma 2 (a) we obtain

$$\begin{aligned} |f(x) - p(x)| &\leq \int_0^x |f'(t) - p'(t)| dt \leq \\ &\leq \mathcal{O} \left\{ \omega \left(F^*, \frac{a}{n} \right) + \left(\frac{\lambda}{\mu} \right)^b \right\} \int_0^x e^{\mu t} dt \leq \\ &\leq \mathcal{O}(e^{\mu x}) \left\{ \omega \left(F^*, \frac{a}{n} \right) + \left(\frac{\lambda}{\mu} \right)^b \right\}. \end{aligned}$$

Hence

$$\begin{aligned} \int_0^{a^r} x^{\alpha} e^{-x} |f(x) - p(x)| dx &\leq \mathcal{O} \left\{ \omega \left(F^*, \frac{a}{n} \right) + \left(\frac{\lambda}{\mu} \right)^b \right\} \int_0^{a^r} x^{\alpha} e^{-(1-\mu)x} dx \leq \\ &\leq \mathcal{O} \left\{ \omega \left(F^*, \frac{a}{n} \right) + \left(\frac{\lambda}{\mu} \right)^b \right\}. \end{aligned}$$

On the other hand we can estimate the integral

$$\int_{a^r}^{\infty} x^{\alpha} e^{-x} |f(x) - p(x)| dx$$

in the following way:

$$\int_{a^r}^{\infty} x^{\alpha} e^{-x} |f(x) - p(x)| dx \leq \int_{a^r}^{\infty} x^{\alpha} e^{-x} (|f(x)| + |p(x)|) dx = I_1 + I_2,$$

where, obviously

$$I_1 = \int_{a^r}^{\infty} x^{\alpha} e^{-x} |f(x)| dx \leq C \int_{a^r}^{\infty} x^{\alpha} e^{-(1-\lambda)x} dx \leq C a^{r\alpha} e^{-(1-\lambda)a^r}.$$

For the estimate of I_2 we have to estimate $|p(x)|$ in case $x \geq a^r$. Using

$$p(x) = \int_{a^r}^x p'(u) du + p(a^r)$$

and taking into account Lemma 1 (a) we obtain

$$|p'(t)| \leq C e^{\mu t}, \quad 0 \leq t \leq a^r,$$

hence

$$|p(t)| \leq c e^{\mu t}, \quad 0 \leq t \leq a^r,$$

consequently

$$|p(x)| \leq \int_{a^r}^x |p'(u)| du + 0 \left(e^{\mu a^r} \right).$$

Taking into account $|p'(u)| \leq c e^{\mu u}$, $0 \leq u \leq a^r$ and using [6] pp. 61–62 we obtain

$$|p'(u)| \leq C e^{\mu u} \left(\frac{2u}{a^r} \right)^{n/r}, \quad a^r \leq u.$$

From this we get

$$\begin{aligned} |p(x)| &\leq c \int_{a^r}^x e^{\mu u} \left(\frac{2u}{a^r} \right)^{\frac{n}{r}} du + c e^{\mu a^r} \leq \\ &\leq c e^{\mu x} \left(\frac{2}{a^r} \right)^{\frac{n}{r}} x^{\frac{n}{r}+1} \frac{r}{n} + c e^{\mu a^r}, \quad x \geq a^r, \end{aligned}$$

so

$$\begin{aligned} I_2 &= \int_{a^r}^{\infty} x^{\alpha} e^{-x} |p(x)| dx = \mathcal{O} \left(e^{\mu a^r} \right) \int_{a^r}^{\infty} x^{\alpha} e^{-x} dx + \\ &\quad + \mathcal{O} \left(\left(\frac{2}{a^r} \right)^{\frac{n}{r}} \cdot \frac{r}{n} \right) \int_{a^r}^{\infty} x^{\alpha + \frac{n}{r} + 1} e^{-(1-\mu)x} dx = \\ &= \mathcal{O} \left(a^{r\alpha} e^{-(1-\mu)a^r} \right) + \mathcal{O} \left(2^{\frac{n}{r}} \frac{r}{n} a^{r\alpha + r} e^{-(1-\mu)a^r} \right), \end{aligned}$$

hence

$$\begin{aligned} & \int_{a^r}^{\infty} x^\alpha e^{-x} |f(x) - p(x)| dx \leq I_1 + I_2 = \\ & = \mathcal{O} \left\{ \omega \left(F^*, \frac{a}{n} \right) + \left(\frac{\lambda}{\mu} \right)^b + a^{r\alpha} e^{-(1-\lambda)a^r} + 2^{\frac{n}{r}} r a^{r\alpha} e^{-(1-\mu)a^r} \right\} = \\ & = \mathcal{O} \left\{ \omega \left(F^*, \frac{a}{n} \right) + \left(\frac{\lambda}{\mu} \right)^b + 2^{\frac{n}{r}} r a^{r\alpha} e^{-(1-\mu)a^r} \right\}, \end{aligned}$$

and Lemma 2 is proved.

Lemma 3 ([3], [4], [5]). *If $g, g' \in C(\lambda)$, $0 < \lambda < \mu < 1$, then*

$$\int_0^{\infty} x^\alpha e^{-x} |F_n(g, x)| dx \leq c \max_{1 \leq k \leq n} |g(x_k)| e^{-\mu x_k} + c \max_{1 \leq k \leq n} |g'(x_k)| e^{-\mu x_k}.$$

Now we return to the PROOF OF THE THEOREM.

Obviously

$$\begin{aligned} \int_0^{\infty} x^\alpha e^{-x} |f(x) - F_n(f, x)| dx & \leq \int_0^{\infty} x^\alpha e^{-x} |f(x) - p(x)| dx + \\ & + \int_0^{\infty} x^\alpha e^{-x} |F_n(f - p, x)| dx, \end{aligned}$$

where $p(x)$ is the polynomial in Lemma 1. Applying Lemma 3 we get

$$\begin{aligned} \int_0^{\infty} x^\alpha e^{-x} |F_n(f - p, x)| dx & \leq c \max_{1 \leq k \leq n} |f(x_k) - p(x_k)| e^{-\mu x_k} + \\ & + c \max_{1 \leq k \leq n} |f'(x_k) - p'(x_k)| e^{-\mu x_k}. \end{aligned}$$

Here $0 \leq x_k \leq 5n$, and applying Lemma 1 (a) we have

$$\max_{1 \leq k \leq n} |f'(x_k) - p'(x_k)| e^{-\mu x_k} = \mathcal{O} \left\{ \omega \left(F^*, \frac{a}{n} \right) + \left(\frac{\lambda}{\mu} \right)^b \right\}.$$

On the other hand, we have seen in the proof of Lemma 2 that

$$|f(x) - p(x)| = \mathcal{O}(e^{\mu x}) \left\{ \omega \left(F^*, \frac{a}{n} \right) + \left(\frac{\lambda}{\mu} \right)^b \right\}, \quad 0 \leq x \leq a^r = \frac{6n}{1-\mu},$$

so

$$\max_{1 \leq k \leq n} |f(x_k) - p(x_k)| = \mathcal{O} \left\{ \omega \left(F^*, \frac{a}{n} \right) + \left(\frac{\lambda}{\mu} \right)^b \right\},$$

hence

$$\int_0^\infty x^\alpha e^{-x} |F_n(f - p, x)| dx = \mathcal{O} \left\{ \omega \left(F^*, \frac{a}{n} \right) + \left(\frac{\lambda}{\mu} \right)^b \right\}.$$

Using this last estimate and applying Lemma 2 we obtain

$$\begin{aligned} & \int_0^\infty x^\alpha e^{-x} |f(x) - F_n(f, x)| dx = \\ &= \mathcal{O} \left\{ \omega \left(F^*, \frac{a}{n} \right) + \left(\frac{\lambda}{\mu} \right)^b + 2^{\frac{n}{r}} r a^{r\alpha} e^{-(1-\mu)a^r} \right\} = \\ &= \mathcal{O} \left\{ \omega \left(F^*, \frac{a}{n} \right) + \left(\frac{\lambda}{\mu} \right)^b + 2^{\frac{n}{r}} r n^\alpha e^{-6n} \right\} \end{aligned}$$

and the Theorem is proved.

Remark.

1. The case $r = 2$ is proved in [4].
2. Our method works also for the Jacobi, Hermite and Markov–Sonin cases.

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