

Structure of normal twisted group rings

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Abstract. Let $K_\lambda G$ be the twisted group ring of a group G over a commutative ring K with 1, and let λ be a factor set (2-cocycle) of G over K . Suppose $f : G \rightarrow U(K)$ is a map from G onto the group of units $U(K)$ of the ring K satisfying $f(1) = 1$. If $x = \sum_{g \in G} \alpha_g u_g \in K_\lambda G$ then we denote $\sum_{g \in G} \alpha_g f(g) u_g^{-1}$ by x^f and assume that the map $x \rightarrow x^f$ is an involution of $K_\lambda G$. In this paper we describe those groups G and commutative rings K for which $K_\lambda G$ is f -normal, i.e. $xx^f = x^f x$ for all $x \in K_\lambda G$.

1. Introduction

Let G be a group and K a commutative ring with unity. Suppose that the elements of the set

$$\Lambda = \{\lambda_{a,b} \in U(K) \mid a, b \in G\}$$

satisfy the condition

$$(1) \quad \lambda_{a,b} \lambda_{ab,c} = \lambda_{b,c} \lambda_{a,bc}$$

for all $a, b, c \in G$. Then Λ will be called a *factor system* (2-cocycle) of the group G over the ring K . The twisted group ring $K_\lambda G$ of G over the commutative ring K is an associative K -algebra with basis $\{u_g \mid g \in G\}$

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and with multiplication defined distributively by $u_g u_h = \lambda_{g,h} u_{gh}$, where $g, h \in G$ and

$$\lambda_{g,h} \in \Lambda = \{\lambda_{a,b} \in U(K) \mid a, b \in G\}.$$

Note that if $\lambda_{g,h} = 1$ for all $g, h \in G$, then $K_\lambda G \cong KG$, where KG is the group ring of the group G over the ring K .

Properties of twisted group algebras and their groups of units were studied by many authors, see, for instance, the paper by S. V. MIHOVSKI and J. M. DIMITROVA [1]. Our aim is to describe the structure of f -normal twisted group rings. This result for group rings was obtained in [2, 3].

We shall refer to two twisted group rings $K_\lambda G$ and $K_\mu G$ as being diagonally equivalent if there exists a map $\theta : G \rightarrow U(K)$ such that

$$\lambda_{a,b} = \theta(a)\theta(b)\mu_{a,b}(\theta(ab))^{-1}.$$

We say that a factor system Λ is normalized if it satisfies the condition

$$\lambda_{a,1} = \lambda_{1,b} = \lambda_{1,1} = 1$$

for all $a, b \in G$.

Hence, given $K_\mu G$ there always exists a diagonally equivalent twisted group ring $K_\lambda G$ with factor system Λ defined by $\lambda_{a,b} = \mu_{1,1}^{-1} \mu_{a,b}$ such that Λ is normalized. From now on, all the factor systems considered are supposed to be normalized.

The map ϕ from the ring $K_\lambda G$ onto $K_\lambda G$ is called an *involution*, if it satisfies the conditions

(i) $\phi(a + b) = \phi(a) + \phi(b)$; (ii) $\phi(ab) = \phi(b)\phi(a)$; (iii) $\phi^2(a) = a$
for all $a, b \in K_\lambda G$.

Let $f : G \rightarrow U(K)$ be a map from the group G onto the group of units $U(K)$ of the commutative ring K , satisfying $f(1) = 1$. For an element $x = \sum_{g \in G} \alpha_g u_g \in K_\lambda G$ we define $x^f = \sum_{g \in G} \alpha_g f(g) u_g^{-1} \in K_\lambda G$.

Let $x \rightarrow x^f$ be an involution of the twisted group ring $K_\lambda G$. The twisted group ring $K_\lambda G$ is called *f-normal* if

$$(2) \quad xx^f = x^f x$$

for all $x \in K_\lambda G$.

Recall that a p -group is called *extraspecial* (see [4], Definition III.13.1) if its centre, commutator subgroup and Frattini subgroup are equal and have order p .

Theorem. Let $x \rightarrow x^f$ be an involution of the twisted group ring $K_\lambda G$. If the ring $K_\lambda G$ is f -normal then the group G and the ring K satisfy one of the following conditions:

- 1) G is abelian and the factor system is symmetric, i.e. $\lambda_{a,b} = \lambda_{b,a}$ for all $a, b \in G$;
- 2) G is an abelian group of exponent 2 and the factor system satisfies

$$(3) \quad (\lambda_{a,b} - \lambda_{b,a})(1 + f(b)\lambda_{b,b}^{-1}) = 0$$

for all $a, b \in G$;

- 3) $G = H \rtimes C_2$ is a semidirect product of an abelian group H of exponent not equal to 2 and $C_2 = \langle a \mid a^2 = 1 \rangle$ with $h^a = h^{-1}$ for all $h \in H$, the factor system of H is symmetric, $f(a) = -\lambda_{a,a}$ and

$$(4) \quad \lambda_{a,h} = f(h)\lambda_{h,h^{-1}}^{-1}\lambda_{h^{-1},a}, \quad \lambda_{h,a} = f(h)\lambda_{h,h^{-1}}^{-1}\lambda_{a,h^{-1}};$$

- 4) G is a hamiltonian 2-group and the factor system satisfies
4.i) for all noncommuting $a, b \in G$

$$(5) \quad \lambda_{a,b} = f(a)\lambda_{a,a^{-1}}^{-1}\lambda_{b,a^{-1}} = f(b)\lambda_{b,b^{-1}}^{-1}\lambda_{b^{-1},a};$$

- 4.ii) $\lambda_{g,h} = \lambda_{h,g}$ for any $h \in C_G(\langle g \rangle)$ and $f(c) = \lambda_{c,c}$ for every c of order 2;

- 5) $G = \Gamma \times C_4$ is a central product of a hamiltonian 2-group Γ and a cyclic group $C_4 = \langle d \mid d^4 = 1 \rangle$ with $\Gamma' = \langle d^2 \rangle$. The factor system satisfies (5) and

$$(6) \quad \lambda_{b,a}\lambda_{ba,d} + f(d)\lambda_{d,d^{-1}}^{-1}\lambda_{a,b}\lambda_{ab,d^{-1}} = 0,$$

where $a, b \in \Gamma$, $a^4 = b^4 = 1$ and $[a, b] \neq 1$;

- 6) G is either $E \times W$ or $(E \times C_4) \times W$, where E is an extraspecial 2-group, $E \times C_4$ is the central product of E and $C_4 = \langle c \mid c^4 = 1 \rangle$ with $E' = \langle c^2 \rangle$ and $\exp(W) \mid 2$. The factor system satisfies:

- 6.i) If $a \in G$ has order 4 then $\lambda_{a,h} = \lambda_{h,a}$ for all $h \in C_G(\langle a \rangle)$;
- 6.ii) if $\langle a, b \rangle$ is a quaternion subgroup of order 8 of G then the properties (5) and (6) are satisfied for every $d \in C_G(\langle a, b \rangle)$ of order 4, and $f(v) = \lambda_{v,v}$ for all $v \in C_G(\langle a, b \rangle)$ of order 2;

6.iii) if $\langle a, b \mid a^4 = b^2 = 1 \rangle$ is the dihedral subgroup of order 8, then $f(b) = -\lambda_{b,b}$ and the properties (4) and (6) are satisfied for every $d \in C_G(\langle a, b \rangle)$ of order 4.

Moreover, the conditions 1)–5) are also sufficient for $K_\lambda G$ to be f -normal. The condition 6) is sufficient if K is an integral domain of characteristic 2.

2. Lemmas

Let C_4 , Q_8 and D_8 be a cyclic group of order 4, a quaternion group of order 8 and a dihedral group of order 8, respectively. As usual, $x^y = y^{-1}xy$, $\exp(G)$ and $C_G(\langle a, b \rangle)$ denote the exponent of G and the centralizer of the subgroup $\langle a, b \rangle$ in G .

It is easy to see that $\lambda_{g,g^{-1}} = \lambda_{g^{-1},g}$ and $u_g^{-1} = \lambda_{g,g^{-1}}^{-1} u_{g^{-1}}$ hold for all $g \in G$.

Lemma 1. *The map $x \rightarrow x^f$ is an involution of the ring $K_\lambda G$ if and only if*

$$f(gh)\lambda_{g,h}^2 = f(g)f(h)$$

for all $g, h \in G$.

PROOF. Let the map $x \rightarrow x^f$ be an involution of the ring $K_\lambda G$. If $g, h \in G$, then $(u_g u_h)^f = u_h^f u_g^f$. Thus

$$\begin{aligned} \lambda_{g,h} f(gh) u_{gh}^{-1} &= (\lambda_{g,h} u_{gh})^f = (u_g u_h)^f = f(g) f(h) u_h^{-1} u_g^{-1} \\ &= f(g) f(h) (\lambda_{g,h}^{-1} u_{gh})^{-1} \end{aligned}$$

and $f(gh)\lambda_{g,h}^2 = f(g)f(h)$ for all $g, h \in G$. $\square$

Clearly, if $K_\lambda G$ is a group ring, then the map $x \rightarrow x^f$ is an involution of the group ring KG if and only if f is a homomorphism from G to $U(K)$.

Lemma 2. *If the ring $K_\lambda G$ is f -normal then the group G satisfies one of the conditions 1)–6) of Theorem 1.*

PROOF. Let $K_\lambda G$ be an f -normal twisted group ring. If $a, b \in G$ and $x = u_a + u_b \in K_\lambda G$, then $x^f = f(a)u_a^{-1} + f(b)u_b^{-1}$ and by (2)

$$\begin{aligned} (7) \quad & f(a)\lambda_{a,a^{-1}}^{-1}\lambda_{a^{-1},b}u_{a^{-1}b} + f(b)\lambda_{b,b^{-1}}^{-1}\lambda_{b^{-1},a}u_{b^{-1}a} \\ &= f(a)\lambda_{a,a^{-1}}^{-1}\lambda_{b,a^{-1}}u_{ba^{-1}} + f(b)\lambda_{b,b^{-1}}^{-1}\lambda_{a,b^{-1}}u_{ab^{-1}}. \end{aligned}$$

Now put $y = u_a(u_1 + u_b)$. Then $y^f = (u_1 + f(b)u_b^{-1})f(a)u_a^{-1}$ and by (2)

$$(8) \quad \lambda_{a,b}u_{ab} + f(b)\lambda_{b,b^{-1}}^{-1}\lambda_{a,b^{-1}}u_{ab^{-1}} = \lambda_{b,a}u_{ba} + f(b)\lambda_{b,b^{-1}}^{-1}\lambda_{b^{-1},a}u_{b^{-1}a}.$$

We shall treat two cases.

- I. Let $[a, b] \neq 1$ for $a, b \in G$ and $a^2 \neq 1$, $b^2 \neq 1$. Then by (8) $b^a = b^{-1}$ and by (7) $a^2 = b^2$. The factor system satisfies

$$(9) \quad \begin{cases} \lambda_{a,b} = f(a)\lambda_{a,a^{-1}}^{-1}\lambda_{b,a^{-1}} = f(b)\lambda_{b,b^{-1}}^{-1}\lambda_{b^{-1},a}; \\ \lambda_{b,a} = f(a)\lambda_{a,a^{-1}}^{-1}\lambda_{a^{-1},b} = f(b)\lambda_{b,b^{-1}}^{-1}\lambda_{a,b^{-1}}. \end{cases}$$

- II. Let $[a, b] \neq 1$ for $a, b \in G$ and $a^2 = 1$, $b^2 \neq 1$. Then by (8) we have $b^a = b^{-1}$ and by (7), $f(a) = -\lambda_{a,a}$. The factor system satisfies

$$\begin{cases} \lambda_{a,b} = f(b)\lambda_{b,b^{-1}}^{-1}\lambda_{b^{-1},a}; \\ \lambda_{b,a} = f(b)\lambda_{b,b^{-1}}^{-1}\lambda_{a,b^{-1}}. \end{cases}$$

Let G be a nonabelian group and let $W = \{g \in G \mid g^2 \neq 1\}$.

First we consider the case when the elements of W commute. Then $\langle w \mid w \in W \rangle$ is an abelian subgroup and if $b \in W$ and $a \in G \setminus \langle W \rangle$ then $a^2=1$ and $(ab)^2 = 1$. Therefore, $b^a = b^{-1}$ for all $b \in W$. Let $c \in C_G(\langle W \rangle) \setminus \langle W \rangle$. Then $c^2 = 1$, $(cb)^2 = 1$ and $cb \notin \langle W \rangle$. But $(cb)^2 = c^2b^2 = 1$ and $b^2 = 1$, which is impossible. Therefore, $C_G(\langle W \rangle) = \langle W \rangle$ and $H = \langle W \rangle$ is a subgroup of index 2. This implies that $G = H \rtimes \langle a \rangle$ and $h^a = h^{-1}$ for all $h \in H$.

Now suppose that in W there exist elements a, b such that $[a, b] \neq 1$. Since $a^2 \neq 1$ and $b^2 \neq 1$, by (I) we have $a^2 = b^2$ and $b^a = b^{-1}$. Then $b^2 = ab^2a^{-1} = b^{-2}$ and the elements a, b are of order 4. Clearly, the subgroup $\langle a, b \rangle$ is a quaternion group of order 8. Let $c \in C_G(\langle a, b \rangle)$. If $c^2 \neq 1$ and $(ac)^2 \neq 1$ then (I) implies that $(ac)^b = (ac)^{-1}$ and $c^2 = 1$, which is impossible. Therefore, if $c \in C_G(\langle a, b \rangle)$ then either $c^2 = 1$ or $c^2 = a^2$.

Let $Q = \langle a, b \rangle$ be a quaternion subgroup of order 8 of G . Then we will prove that $G = Q \cdot C_G(Q)$. Suppose $g \in G \setminus C_G(Q)$. Pick the elements $a, b \in Q$ of order 4 such that $a^g = a^{-1}$ and $b^g = b^{-1}$. Then $(ab)^g = ab$ and $d = gab \in C_G(Q)$. It follows that $g = d(ab)^{-1}$ and $G = Q \cdot C_G(Q)$. Similary as in [3] we obtain that G satisfies the conditions 4) or 5) of the Theorem. $\square$

3. Proof of Theorem

Necessity. Let $K_\lambda G$ be f -normal. Then by Lemma 2 G satisfies one of the conditions 1)–5) of the Theorem.

First, suppose that G is abelian of exponent greater than 2 and $a, b \in G$. If $b^2 \neq 1$ then by (8) we have $\lambda_{a,b} = \lambda_{b,a}$.

Let a, b be elements of order two and assume that there exists c with $c^2 = a$. Then by (1) we have

$$(10) \quad \lambda_{c^2,b}\lambda_{c,c} = \lambda_{c,cb}\lambda_{c,b} \quad \text{and} \quad \lambda_{b,c^2}\lambda_{c,c} = \lambda_{bc,c}\lambda_{b,c}.$$

Since $c^2 \neq 1$, we have $\lambda_{c,cb} = \lambda_{bc,c}$ and $\lambda_{c,b} = \lambda_{b,c}$. Then (10) implies $\lambda_{c^2,b} = \lambda_{b,c^2}$ and $\lambda_{a,b} = \lambda_{b,a}$.

Let $a^2 = b^2 = 1$ such that neither a nor b is the square of any element of G . Then there exists c such that $(ca)^2 \neq 1$. Thus,

$$(11) \quad \lambda_{ca,b}\lambda_{c,a} = \lambda_{c,ab}\lambda_{a,b}, \quad \lambda_{b,ac}\lambda_{a,c} = \lambda_{ba,c}\lambda_{b,a}.$$

Since $\lambda_{b,ac} = \lambda_{ac,b}$ and $\lambda_{c,a} = \lambda_{a,c}$ from (11) we have $\lambda_{a,b} = \lambda_{b,a}$ for all $a, b \in G$. Therefore, if G is abelian and $G^2 \neq 1$ then the factor system is symmetric and $K_\lambda G$ is commutative.

Now, let $\exp(G) = 2$. Then by (8) $\lambda_{a,b} + f(b)\lambda_{b,b}^{-1}\lambda_{a,b} = \lambda_{b,a} + f(b)\lambda_{b,b}^{-1}\lambda_{b,a}$ for all $a, b \in G$. Therefore, $(\lambda_{a,b} - \lambda_{b,a})(1 + f(b)\lambda_{b,b}^{-1}) = 0$.

Next, let $G = H \rtimes C_2$ be a semidirect product of an abelian group H with $\exp(H) \neq 2$ and $C_2 = \langle a \mid a^2 = 1 \rangle$, and with $h^a = h^{-1}$ for all $h \in H$. Clearly, $K_\lambda H$ is f -normal and the factor system of H is symmetric. Put $x = u_h + u_a$ for $h \in H$. Since $K_\lambda G$ is f -normal, we have $S_f(x) = xx^f - x^f x = 0$ and

$$(12) \quad \begin{aligned} & f(a)\lambda_{a,a}^{-1}\lambda_{h,a}u_{ha} + f(h)\lambda_{h,h^{-1}}^{-1}\lambda_{a,h^{-1}}u_{ah^{-1}} \\ & - f(h)\lambda_{h,h^{-1}}^{-1}\lambda_{h^{-1},a}u_{h^{-1}a} - f(a)\lambda_{a,a}^{-1}\lambda_{a,h}u_{ah} = 0. \end{aligned}$$

We will prove $u_a u_h = u_h^f u_a$ for every $h \in H$.

First, let $h^2 \neq 1$. Because $h^a = h^{-1}$, by (12) we have

$$(13) \quad u_a^f u_h + u_h^f u_a = 0$$

and

$$(14) \quad \begin{cases} f(a)\lambda_{a,a}^{-1}\lambda_{a,h} + f(h)\lambda_{h,h^{-1}}^{-1}\lambda_{h^{-1},a} = 0; \\ f(a)\lambda_{a,a}^{-1}\lambda_{h,a} + f(h)\lambda_{h,h^{-1}}^{-1}\lambda_{a,h^{-1}} = 0. \end{cases}$$

Now, let $h^2 = 1$. Then there exists $b \in H$ with $b^2 \neq 1$ and $(hb)^2 \neq 1$. Put $x = u_a + u_h u_b$. Because $(hb)^a = (hb)^{-1}$ and $S_f(x) = xx^f - x^f x = 0$ we have

$$(15) \quad u_a^f u_h u_b + (u_h u_b)^f u_a = 0.$$

Since $[u_h, u_b] = 1$, by (15) and (13) we have $u_a^f(u_h u_b) = u_a^f u_b u_h = -u_b^f u_a u_h$ and $u_a^f(u_h u_b) = -(u_h u_b)^f u_a = -u_b^f u_h^f u_a$. Therefore, $u_a u_h = u_h^f u_a$ for all $h \in H$ and this implies

$$\begin{cases} \lambda_{a,h} = f(h)\lambda_{h,h^{-1}}^{-1}\lambda_{h^{-1},a}; \\ \lambda_{h,a} = f(h)\lambda_{h,h^{-1}}^{-1}\lambda_{a,h^{-1}}, \end{cases}$$

and, by (14), $f(a) = -\lambda_{a,a}$.

Let G be a hamiltonian 2-group. It is well known (see [5], Theorem 12.5.4) that $G = Q_8 \times W$, where Q_8 is a quaternion group and $\exp(W)|2$. If $a, b \in G$ are noncommuting elements of order 4, then $a^b = a^{-1}$ and by (8) we have 4.i) of the theorem. If $c, d \in G$ are involutions, then c and d commute with all $a \in G$ of order 4. Then $H = \langle a, d, c \rangle$ is abelian of exponent greater than 2 and $K_\lambda H$ is f -normal. By the condition 1) of the theorem, the factor system of H is symmetric, and u_a and u_b commute with u_c .

Now prove $f(c) = \lambda_{c,c}$ for all involutions $c \in G$. Choose the elements a, b of order 4 such that $b^a = b^{-1}$. Put $x = u_c u_a + u_b$. Since $\lambda_{a,c} = \lambda_{c,a}$ and $\lambda_{b,c} = \lambda_{c,b}$ by (2), for x we obtain

$$\begin{aligned} S_f(x) &= (f(b)u_a u_b^{-1} + f(a)f(c)\lambda_{c,c}^{-1}u_b u_a^{-1} \\ &\quad - f(b)u_b^{-1}u_a - f(a)f(c)\lambda_{c,c}^{-1}u_a^{-1}u_b)u_c = 0 \end{aligned}$$

and $f(b)\lambda_{b,b^{-1}}^{-1}\lambda_{a,b^{-1}} = f(c)f(a)\lambda_{c,c}^{-1}\lambda_{a,a^{-1}}^{-1}\lambda_{a^{-1},b}$. From this property and (9) we deduce $f(c) = \lambda_{c,c}$.

Now, suppose that either $G = E \times W$ or $G = (E \text{ Y } C_4) \times W$, where E is an extraspecial 2-group, $\exp(W)|2$ and $E \text{ Y } C_4$ is the central product of E and $C_4 = \langle c \rangle$ with $E' = \langle c^2 \rangle$.

Let a be an element of order 4 and $h \in C_G(\langle a \rangle)$. Then by the condition 1) of the theorem $\lambda_{a,h} = \lambda_{h,a}$.

Let $\langle a, b \mid a, b \in G \rangle$ be the quaternion subgroup of order 8. Then by 4) we obtain (5).

Now, let $G = \langle a, b \rangle \rtimes \langle d \mid d^4 = 1 \rangle$ be a subgroup of G and $d^2 = a^2$. Then $a^b = a^{-1}$, and $\langle a, d \rangle$ and $\langle b, d \rangle$ are abelian subgroups of exponent not equal to 2 and by the condition 1) of the theorem, $\lambda_{a,d} = \lambda_{d,a}$ and $\lambda_{b,d} = \lambda_{d,b}$. Put $x = u_b + u_a u_d$. Since $K_\lambda G$ is f -normal, we obtain

$$\begin{aligned} & f(b)\lambda_{b,b^{-1}}^{-1}\lambda_{a,b^{-1}}u_{ab^{-1}}u_d + f(d)f(a)\lambda_{a,a^{-1}}^{-1}\lambda_{d,d^{-1}}^{-1}\lambda_{b,a^{-1}}u_{ba^{-1}}u_{d^{-1}} \\ &= f(d)f(a)\lambda_{a,a^{-1}}^{-1}\lambda_{d,d^{-1}}^{-1}\lambda_{a^{-1},b}u_{a^{-1}b}u_{d^{-1}} + f(b)\lambda_{b,b^{-1}}^{-1}\lambda_{b^{-1},a}u_{b^{-1}a}u_d \end{aligned}$$

and by (5)

$$\begin{aligned} & \lambda_{b,a}\lambda_{ab^{-1},d}u_{ab^{-1}d} + f(d)\lambda_{d,d^{-1}}^{-1}\lambda_{a,b}\lambda_{ba^{-1},d^{-1}}u_{ba^{-1}d^{-1}} \\ &= f(d)\lambda_{d,d^{-1}}^{-1}\lambda_{b,a}\lambda_{a^{-1}b,d^{-1}}u_{a^{-1}bd^{-1}} + \lambda_{a,b}\lambda_{b^{-1}a,d}u_{b^{-1}ad}. \end{aligned}$$

Since $d^2 \in G'$ and $a^2 = b^2$, we have $a^{-1}bd^{-1} = abd$, $ab^{-1}d = ba^{-1}d^{-1}$ and

$$\lambda_{b,a}\lambda_{ba,d} + f(d)\lambda_{d,d^{-1}}^{-1}\lambda_{a,b}\lambda_{ab,d^{-1}} = 0.$$

Therefore, we proved 6.i).

If $\langle a, b \mid a^4 = b^2 = 1 \rangle$ is the dihedral subgroup of order 8 of G , then by 3) of the theorem we have (4) and $f(b) = -\lambda_{b,b}$.

Let $L = D_8 \rtimes C_4 = \langle a, b \mid a^4 = b^2 = 1 \rangle \rtimes \langle c \rangle$. Then any $x \in K_\lambda L$ can be written as $x = x_0 + x_1 u_c$, where $x_0, x_1 \in K_\lambda D_8$. Since $K_\lambda G$ is f -normal, $K_\lambda L$ is f -normal, too, and $(x_0^f x_1 - x_1 x_0^f)u_c = (x_0 x_1^f - x_1^f x_0)u_c^f$. By the f -normality of $K_\lambda D_8$ $(x_0 + x_1)(x_0 + x_1)^f = (x_0 + x_1)^f(x_0 + x_1)$ and we have

$$(x_0^f x_1 - x_1 x_0^f)u_c - (x_0 x_1^f - x_1^f x_0)u_c^f = (x_0^f x_1 - x_1 x_0^f)(u_c - u_c^f).$$

If $x_0^f x_1 - x_1 x_0^f$ can be written as a sum of elements of form $u_a^f u_b - u_b u_a^f$ then

$$\begin{aligned} & (x_0^f x_1 - x_1 x_0^f)(u_c - u_c^f) = (\lambda_{b,a}\lambda_{ba,c} + f(c)\lambda_{c,c^{-1}}^{-1}\lambda_{a,b}\lambda_{ab,c^{-1}})u_{bac} \\ & - (\lambda_{a,b}\lambda_{ab,c} + f(c)\lambda_{c,c^{-1}}^{-1}\lambda_{b,a}\lambda_{ba,c^{-1}})u_{abc} = 0 \end{aligned}$$

and we have (6).

Sufficiency. We wish to prove that $S_f(x) = xx^f - x^fx$ is equal to 0 for all $x \in K_\lambda G$. Let $x = \sum_{g \in G} \alpha_g u_g \in K_\lambda G$. It is easy to see that $S_f(x)$ is a sum of elements of the form

$$\begin{aligned} S_f(g, h) &= \alpha_g \alpha_h (f(h) \lambda_{h, h^{-1}}^{-1} \lambda_{g, h^{-1}} u_{gh^{-1}} + f(g) \lambda_{g, g^{-1}}^{-1} \lambda_{h, g^{-1}} u_{hg^{-1}} \\ &\quad - f(h) \lambda_{h, h^{-1}}^{-1} \lambda_{h^{-1}, g} u_{h^{-1}g} - f(g) \lambda_{g, g^{-1}}^{-1} \lambda_{g^{-1}, h} u_{g^{-1}h}). \end{aligned}$$

First, let G be abelian of exponent greater than 2, and assume that the factor system of G is symmetric. Then $K_\lambda G$ is commutative, and therefore, f -normal.

Next, suppose that G is of exponent 2 and the factor system satisfies $(\lambda_{g, h} - \lambda_{h, g})(1 + f(h) \lambda_{h, h}^{-1}) = 0$ for all $g, h \in G$.

This implies $(\lambda_{g, h} - \lambda_{h, g})(f(g) \lambda_{g, g}^{-1} - f(h) \lambda_{h, h}^{-1}) = 0$ for all $g, h \in G$. Then

$$\begin{aligned} S_f(g, h) &= \alpha_g \alpha_h (f(h) \lambda_{h, h}^{-1} \lambda_{g, h} u_{gh} + f(g) \lambda_{g, g}^{-1} \lambda_{h, g} u_{hg} - f(h) \lambda_{h, h}^{-1} \lambda_{h, g} u_{hg} \\ &\quad - f(g) \lambda_{g, g}^{-1} \lambda_{g, h} u_{gh}) = \alpha_g \alpha_h (f(h) \lambda_{h, h}^{-1} - f(g) \lambda_{g, g}^{-1})(\lambda_{g, h} - \lambda_{h, g}) u_{gh} = 0 \end{aligned}$$

and $S_f(x) = 0$, thus, $K_\lambda G$ is f -normal.

Now, let $G = H \rtimes C_2$, where H is an abelian group of exponent not equal to 2 and $C_2 = \langle a \rangle$ with $h^a = h^{-1}$ for all $h \in H$. Using the properties of the factor system we obtain

$$(16) \quad \begin{aligned} f(a) u_a^{-1} u_h &= -f(h) u_h^{-1} u_a, & f(a) u_h u_a^{-1} &= -f(h) u_a u_h^{-1}, \\ u_a^f y &= -y^f u_a, & y u_a^f &= -u_a y^f \end{aligned}$$

for any $h \in H$ and $y \in K_\lambda H$. If $x = x_1 + x_2 u_a \in K_\lambda G$ where $x_1, x_2 \in K_\lambda H$, then $x^f = x_1^f + f(a) u_a^{-1} x_2^f$ and

$$xx^f = x_1 x_1^f + f(a) x_1 u_a^{-1} x_2^f + x_2 u_a x_1^f + f(a) x_2 x_2^f.$$

Because in $K_\lambda H$ the factor system is symmetric and $K_\lambda H$ is commutative, by (16) we have

$$xx^f = x_1 x_1^f + (x_2 x_1 - x_1 x_2) u_a + f(a) x_2 x_2^f = x_1 x_1^f + f(a) x_2 x_2^f.$$

Similarly, $x^f x = x_1^f x_1 + f(a) x_2^f x_2$ and we conclude that $S_f(x) = 0$ and $K_\lambda G$ is f -normal.

Next, let G be a hamiltonian 2-group. Then $G = Q_8 \times W$, where $Q_8 = \langle a, b \rangle$ is a quaternion group and $\exp(W)|2$. Suppose that the conditions 4.i)–4.ii) of the theorem are satisfied. If $H = \langle a^2, W \rangle$ then any element $x \in K_\lambda G$ can be written as

$$x = x_0 + x_1 u_a + x_2 u_b + x_3 u_{ab},$$

where $x_i \in K_\lambda H$, ($i = 0, \dots, 3$). Since $\langle a \rangle \times H$ and $\langle b \rangle \times H$ are abelian groups of exponent 4, by the condition 1) of the theorem the elements x_0, x_1, x_2, x_3 commute with u_a, u_b and u_{ab} . Since $K_\lambda H$ is f -normal, we have $x_i x_j^f - x_i^f x_j = x_j^f x_i - x_j x_i^f$. Using these properties we obtain

$$\begin{aligned} S_f(x) &= (x_1 x_2^f - x_1^f x_2)(\lambda_{b,a} u_{ba} - \lambda_{a,b} u_{ab}) \\ &\quad + (x_1 x_3^f - x_1^f x_3)(\lambda_{ab,a} u_b - \lambda_{a,ab} u_{b^3}) \\ &\quad + (x_2 x_3^f - x_2^f x_3)(\lambda_{ab,b} u_{a^3} - \lambda_{b,ab} u_a). \end{aligned}$$

Clearly, the element $x_i x_j^f - x_i^f x_j$ can be written as a sum of elements of form

$$S_f(c, d) = \gamma_{c,d}(f(d)u_c u_d^{-1} - f(c)u_c^{-1} u_d),$$

where $c, d \in H$. Since H is an elementary 2-subgroup, by the condition 4.ii) $f(d) = \lambda_{d,d}$, $f(c) = \lambda_{c,c}$, and we obtain

$$S_f(c, d) = \gamma_{c,d}(f(d)\lambda_{d,d}^{-1}\lambda_{c,d}u_{cd} - f(c)\lambda_{c,c}^{-1}\lambda_{c,d}u_{cd}) = 0.$$

Therefore, $S_f(x) = 0$ and $K_\lambda G$ is f -normal.

Next, let $G = H \times W$, where H is an extraspecial 2-group and $\exp(W)|2$. Since G is a locally finite group, it suffices to establish the f -normality of all finite subgroups H of G . Let G be a finite group and $G = H \times W$, where H is a finite extraspecial 2-group and $\exp(W)|2$. We know (see [4], Theorem III.13.8) that H is a central product of n copies of dihedral groups of order 8 or a central product of a quaternion group of order 8 and $n-1$ copies of dihedral groups of order 8. We can write $H_n = H$. Then $G = H_n \times W$ and by induction on n we prove the f -normality of $K_\lambda G$.

If $n = 1$ then either $H_1 = Q_8$ or $H_1 = D_8$ or $H_1 = Q_8 \times C_4$. In the first and second cases the f -normality $K_\lambda G$ is implied by the conditions 3) or 4) of the theorem.

Let $G = Q_8 \amalg C_4$. Then any element $x \in K_\lambda G$ can be written as $x = x_0 + x_1 u_c$, where $x_i \in K_\lambda Q_8$, $c \in C_4$ and $c^2 \in Q_8$. From the f -normality of $K_\lambda Q_8$ we obtain $x_0^f x_1 - x_1 x_0^f = x_1^f x_0 - x_0 x_1^f$ and $S_f(x) = (x_0^f x_1 - x_1 x_0^f)(u_c - u_c^f)$. The element $x_0^f x_1 - x_1 x_0^f$ can be written as a sum of elements of form $\alpha(u_a^f u_b - u_b u_a^f)$, where $\alpha \in K$, $a, b \in Q_8$. We will prove $S_f(a, b) = (u_a^f u_b - u_b u_a^f)(u_c - u_c^f) = 0$ for all $a, b \in Q_8$.

If $a, b \in Q_8$ does not generate Q_8 then $u_a u_b = u_b u_a$ and $S_f(a, b) = 0$. Let $\langle a, b \rangle = Q_8$. Then by (5)

$$\begin{aligned} S_f(a, b) &= (\lambda_{b,a} u_{ba} - \lambda_{a,b} u_{ab})(u_c - u_c^f) \\ &= (\lambda_{b,a} \lambda_{ba,c} + f(c) \lambda_{c,c^{-1}}^{-1} \lambda_{a,b} \lambda_{ab,c^{-1}}) u_{bac} \\ &\quad + (\lambda_{a,b} \lambda_{ab,c} + f(c) \lambda_{c,c^{-1}}^{-1} \lambda_{b,a} \lambda_{ba,c^{-1}}) u_{abc} \end{aligned}$$

and from (6) $S_f(a, b) = 0$.

It is easy to see $D_8 \amalg D_8 \cong Q_8 \amalg Q_8$, and H_n ($n > 1$) can be written as $Q_8 \amalg H_{n-1}$.

Let $Q_8 = \langle a, b \rangle$ and $L = W \times H_{n-1}$. Then any element $x \in K_\lambda G$ can be written as

$$x = x_0 + x_1 u_a + x_2 u_b + x_3 u_a u_b,$$

where $x_i \in K_\lambda L$. By 6.i) the x_i commute with u_a and u_b . Since $\langle a, b \rangle$ is a quaternion group of order 8, by the condition 6.ii) of the theorem we have $u_a u_b = u_b^f u_a = u_b u_a^f$. Hence,

$$\begin{aligned} S_f(x) &= (x_0 x_1^f - x_1^f x_0) u_a^f + (x_0 x_2^f - x_2^f x_0) u_b^f + (x_0 x_3^f - x_3^f x_0) u_b^f u_a^f \\ &\quad + (x_1 x_0^f - x_0^f x_1) u_a + (x_1 x_2^f - x_2^f x_1) u_a u_b^f + (x_1 x_3^f - x_3^f x_1) u_b f(a) \\ (17) \quad &\quad + (x_2 x_0^f - x_0^f x_2) u_b + (x_2 x_1^f - x_1^f x_2) u_a u_b + (x_2 x_3^f - x_3^f x_2) u_a^f f(b) \\ &\quad + (x_3 x_0^f - x_0^f x_3) u_a u_b + (x_3 x_1^f - x_1^f x_3) u_a u_{ab} \\ &\quad + (x_3 x_2^f - x_2^f x_3) u_a f(b). \end{aligned}$$

Since by induction $K_\lambda L$ is f -normal, $(x_i + x_j)(x_i + x_j)^f = (x_i + x_j)^f (x_i + x_j)$ implies $x_i x_j^f - x_i^f x_j = x_j^f x_i - x_j x_i^f$ and $x_i x_j^f - x_j^f x_i =$

$x_i^f x_j - x_j x_i^f$. Therefore, by (17)

$$\begin{aligned} S_f(x) &= (x_0 x_1^f - x_1^f x_0)(u_a^f - u_a) + (x_0 x_2^f - x_2^f x_0)(u_b^f - u_b) \\ &\quad + (x_0 x_3^f - x_3^f x_0)(u_a^f - u_a)u_b + (x_1 x_2^f - x_1^f x_2)u_a(u_b^f - u_b) \\ &\quad + (x_1 x_3^f - x_1^f x_3)u_a(u_b - u_b^f)f(a) + (x_2 x_3^f - x_2^f x_3)(u_a^f - u_a)f(b). \end{aligned}$$

Clearly, the element $x_i x_j^f - x_j^f x_i$ can be written as a sum of elements of form $S_f(c, d) = \gamma_{c,d}(u_c u_d^f - u_d^f u_c)$, where $c, d \in L$, $\gamma_{c,d} \in K$. We will prove $S_f(c, d, a) = (u_c u_d^f - u_d^f u_c)(u_a - u_a^f) = 0$ for any $c, d \in L$.

We consider the following cases:

Case 1). Let $[c, d] = 1$. Then $L = \langle c, d, a \rangle$ is abelian with $\exp(L) \neq 2$, and by 6.i) the factor system is symmetric and $S_f(c, d, a) = 0$.

Case 2). Let $\langle c, d \rangle = Q_8$. Then by 6.ii) (5) holds and $(u_c u_d^f - u_d^f u_c)(u_a - u_a^f) = (\lambda_{d,c} \lambda_{dc,a} + f(a) \lambda_{a,a^{-1}}^{-1} \lambda_{c,d} \lambda_{cd,a^{-1}}) u_{dca} - (\lambda_{c,d} \lambda_{cd,a} + f(a) \lambda_{a,a^{-1}}^{-1} \lambda_{d,c} \lambda_{a^{-1},dc}) u_{cda}$. Now by 6.ii) the property (6) is satisfied and we conclude $S_f(c, d, a) = 0$.

Case 3). Let $\langle c, d \rangle = D_8$ and $c^4 = d^2 = 1$. Then by 6.iii) $f(d) = -\lambda_{d,d}$ and by (4)

$$\begin{aligned} (u_c u_d^f - u_d^f u_c)(u_a - u_a^f) &= (\lambda_{d,c} u_{dc} - \lambda_{c,d} u_{cd})(u_a - u_a^f) \\ &= (\lambda_{c,d} \lambda_{cd,a} + f(a) \lambda_{a,a^{-1}}^{-1} \lambda_{dc,a^{-1}} \lambda_{d,c}) u_{cda} \\ &\quad + (\lambda_{d,c} \lambda_{dc,a} + f(a) \lambda_{a,a^{-1}}^{-1} \lambda_{cd,a^{-1}} \lambda_{c,d}) u_{dca}. \end{aligned}$$

Now by 6.ii) we have (6) and we conclude $S_f(c, d, a) = 0$.

Case 4). Let $\langle c, d \rangle = D_8$ and $d^4 = c^2 = 1$. Then by (4)

$$\begin{aligned} u_c u_d^f - u_d^f u_c &= f(d) \lambda_{d,d^{-1}}^{-1} \lambda_{c,d^{-1}} u_{dc} - f(d) \lambda_{d,d^{-1}}^{-1} \lambda_{d^{-1},c} u_{cd} \\ &= \lambda_{d,c} u_{dc} - \lambda_{c,d} u_{cd}. \end{aligned}$$

Similarly to the case 3) we have $S_f(c, d, a) = 0$.

Case 5). Let $\langle c, d \rangle = D_8$ and $d^2 = c^2 = 1$. Then by 6.iii) $f(d) = -\lambda_{d,d}$. In $\langle c, d \rangle$ we choose a new generator system $\{a_1, b_1 \mid a_1^4 = b_1^2 =$

$1, a_1^{b_1} = a_1^{-1}\}$ such that $c = b_1$ and $d = a_1^i b_1$, where $i = 1$ or 3 . Then $a^2 = a_1^2$ and

$$\begin{aligned} (u_c u_d^f - u_d^f u_c)(u_a - u_a^f) &= (u_d u_c - u_c u_d)(u_a - u_a^f) \\ &= \lambda_{a_1^i, b_1}^{-1} (u_{a_1^i} u_{b_1} - u_{b_1} u_{a_1^i})(u_a - u_a^f) u_{b_1}. \end{aligned}$$

As in the Case 3) it is easy to see $(u_{a_1^i} u_{b_1} - u_{b_1} u_{a_1^i})(u_a - u_a^f) = 0$ and $S_f(c, d, a) = 0$.

Analogously, the element $x_i x_j^f - x_i^f x_j$ can be written as a sum of elements of form $\gamma_{c,d}(u_c u_d^f - u_d^f u_c)$, where $c, d \in L$. Let us prove that if $c, d \in L$, then $S_f(c, d, a) = (u_c u_d^f - u_d^f u_c)(u_a - u_a^f) = 0$.

Let $z \in L$, $a \in Q_8$ be commuting elements of order 4 with $z^2 = a^2$. First, we will prove that K is of characteristic 2, then $(u_z + u_z^f)(u_a + u_a^f) = 0$.

Indeed,

$$\begin{aligned} (u_z + u_z^f)(u_a + u_a^f) &= (\lambda_{z,a} + f(z)\lambda_{z,z^{-1}}^{-1}f(a)\lambda_{a,a^{-1}}^{-1}\lambda_{z^{-1},a^{-1}})u_{za} \\ &\quad + (f(a)\lambda_{a,a^{-1}}^{-1}\lambda_{z,a^{-1}} + f(z)\lambda_{z,z^{-1}}^{-1}\lambda_{z^{-1},a})u_{za^3}. \end{aligned}$$

First let za be a noncentral element of order 2. Then by 6.iii) $f(za) = \lambda_{za,za}$. Since $((u_z u_a)u_a)u_{a^3} = u_z(u_a(u_a u_{a^3}))$ we conclude that

$$\lambda_{z,a}\lambda_{za,a}\lambda_{za^2,a^3} = \lambda_{z,a}\lambda_{a,1}\lambda_{a,a^{-1}}$$

and $\lambda_{a,a^{-1}}^{-1} = \lambda_{z^3,a^3}^{-1}\lambda_{za,a}^{-1}$. Clearly, $f(z)f(a) = f(za)\lambda_{z,a}^2 = \lambda_{za,za}\lambda_{z,a}^2$ and

$$\begin{aligned} (18) \quad &\lambda_{z,a} + f(z)\lambda_{z,z^{-1}}^{-1}f(a)\lambda_{a,a^{-1}}^{-1}\lambda_{z^{-1},a^{-1}} \\ &= \lambda_{z,a}(1 + (\lambda_{za,az}\lambda_{a,z})\lambda_{z,z^{-1}}^{-1}\lambda_{a,a^{-1}}^{-1}\lambda_{z^{-1},a^{-1}}) \\ &= \lambda_{z,a}(1 + \lambda_{z,za^2}\lambda_{a,az}\lambda_{a,a^{-1}}^{-1}\lambda_{z,za^2}^{-1}\lambda_{z^{-1},a^{-1}}) \\ &= \lambda_{z,a}(1 + (\lambda_{a,az}\lambda_{zaa,a^{-1}})\lambda_{a,a^{-1}}^{-1}) \\ &= \lambda_{z,a}(1 + \lambda_{za,aa^{-1}}\lambda_{a,a^{-1}}\lambda_{a,a^{-1}}^{-1}) = 2\lambda_{z,a} = 0. \end{aligned}$$

By (1) we have

$$\begin{aligned} &(\lambda_{z,a^{-1}}\lambda_{za^{-1},za^{-1}})\lambda_{z^{-1},a} = \lambda_{z,a^{-1}za^{-1}}\lambda_{a^{-1},z^{-1}}\lambda_{z^{-1},a} \\ &= \lambda_{z,z^{-1}}(\lambda_{a^{-1},az^{-1}}\lambda_{a,z^{-1}}) = \lambda_{z^{-1},z}\lambda_{aa^{-1},z^{-1}}\lambda_{a,a^{-1}} = \lambda_{z^{-1},z}\lambda_{a,a^{-1}} \end{aligned}$$

and since az^{-1} has order 2, $f(az^{-1}) = \lambda_{az^{-1}, az^{-1}}$, and we obtain

$$\begin{aligned}
 & f(a^{-1})^{-1}f(a^{-1})(f(a)\lambda_{a,a^{-1}}^{-1}\lambda_{z,a^{-1}} + f(z)\lambda_{z,z^{-1}}^{-1}\lambda_{z^{-1},a}) \\
 &= f(a^{-1})^{-1}(\lambda_{a,a^{-1}}^2\lambda_{a,a^{-1}}^{-1}\lambda_{z,a^{-1}} + f(az^{-1})\lambda_{a^{-1},z}^2\lambda_{z,z^{-1}}^{-1}\lambda_{z^{-1},a}) \\
 (19) \quad &= f(a^{-1})^{-1}(\lambda_{a,a^{-1}}\lambda_{z,a^{-1}} + \lambda_{z,a^{-1}}(\lambda_{z,a^{-1}}\lambda_{az^{-1},az^{-1}}\lambda_{z^{-1},a})\lambda_{z,z^{-1}}^{-1}) \\
 &= f(a^{-1})^{-1}(\lambda_{z,a^{-1}}(\lambda_{a^{-1},a} - \lambda_{z,z^{-1}}\lambda_{a^{-1},a}\lambda_{z^{-1},z}^{-1}\lambda_{z^{-1},a})) \\
 &= 2f(a^{-1})^{-1}\lambda_{z,a^{-1}}\lambda_{a,a^{-1}} = 0.
 \end{aligned}$$

Clearly, if $[c, d] = 1$ then $S_f(c, d, a)$ can be written as

$$\begin{aligned}
 (20) \quad & S_f(c, d, a) = (u_c u_d^f + (u_d^f u_c)^f)(u_a - u_a^f) \\
 &= f(d)\lambda_{d,d^{-1}}\lambda_{c,d^{-1}}(u_{cd^{-1}} - u_{cd^{-1}}^f)(u_a - u_a^f).
 \end{aligned}$$

Similarly, the element $x_i x_j^f - x_i^f x_j$ can be written as a sum of elements of form $\gamma_{c,d}(u_c u_d^f - u_c^f u_d)$, where $c, d \in L$. Now let us prove $S_f(c, d, a) = (u_c u_d^f - u_c^f u_d)(u_a - u_a^f) = 0$, where $c, d \in L$.

We consider the following cases:

Case 1). Let $[c, d] = 1$, $c^2 = d^2 = 1$ and $c, d \notin \zeta(G)$. Then $S = \langle c, d, a \rangle$ is abelian of exponent greater than 2 and by 6.i) the factor system of S is symmetric. We know that in L every element of order 2 is either central or coincides with a noncentral element of some dihedral subgroup of order 8. Since $c, d \notin \zeta(G)$, we have $f(c) = \lambda_{c,c}$ and $f(d) = \lambda_{d,d}$ and

$$S_f(c, d, a) = \lambda_{c,d}(f(d)\lambda_{d,d}^{-1} - f(c)\lambda_{c,c}^{-1})u_{cd}(u_a - u_a^f) = 0.$$

Case 2). Let $[c, d] = 1$, $c^2 = d^2 = 1$ and $c, d \in \zeta(G)$. Then $c = d = a^2$ and $S_f(c, d, a) = 0$.

Case 3). Let $[c, d] = 1$, $c^2 = d^2 = 1$ and $c \in \zeta(G)$, $d \notin \zeta(G)$. Then $f(d) = \lambda_{d,d}^{-1}$, $c = a^2$ and

$$\begin{aligned}
 S_f(c, d, a) &= -u_d(u_{a^2} + u_{a^2}^f)(u_a - u_a^f) \\
 &= -u_d(\lambda_{a,a^2}u_{a^{-1}} - f(a)\lambda_{a,a^{-1}}^{-1}u_a)(1 + f(a^2)\lambda_{a^2,a^2}^{-1}).
 \end{aligned}$$

Since K is an integral domain of characteristic 2 and $f^2(a^2) = \lambda_{a^2,a^2}^2 f(a^4) = \lambda_{a^2,a^2}^2$, we conclude $f(a^2) = \pm \lambda_{a^2,a^2}$ and $S_f(c, d, a) = 0$.

Case 4). Let $[c, d] = 1$, $d^2 = 1$ and suppose that c has order 4. Then dc has order 4 and by (20) $S_f(c, d, a) = 0$.

Case 5). Let $[c, d] = 1$ with c, d of order 4. Then $d^2 = c^2 = a^2$,

$$S_f(c, d, a) = (f(d)\lambda_{d,d^{-1}}^{-1}\lambda_{c,d^{-1}} + f(c)\lambda_{c,c^{-1}}^{-1}\lambda_{c^{-1},d})u_{cd^{-1}}(u_a - u_a^f),$$

and by (19) we have $S_f(c, d, a) = 0$.

Case 6). Let $\langle c, d \rangle$ be a quaternion group of order 8. Then by 6.ii) (5) holds and

$$\begin{aligned} u_c u_d^f - u_c^f u_d &= (f(d)\lambda_{d,d^{-1}}^{-1}\lambda_{c,d^{-1}} - f(c)\lambda_{c,c^{-1}}^{-1}\lambda_{c^{-1},d})u_{c^{-1}d} \\ &= (\lambda_{d,c} - \lambda_{d,c})u_{c^{-1}d} = 0. \end{aligned}$$

Case 7). Let $\langle c, d \rangle$ be a dihedral group of order 8. If $c^2 \neq 1$ then $f(d) = \lambda_{d,d}$ and

$$\begin{aligned} S_f(c, d, a) &= (\lambda_{c,d}u_{cd} + f(c)\lambda_{c,c^{-1}}^{-1}\lambda_{c^{-1},d}u_{dc})(u_a - u_a^f) \\ &= (\lambda_{c,d}u_{cd} + \lambda_{d,c}u_{dc})(u_a - u_a^f) = (\lambda_{c,d}\lambda_{cd,a} + f(a)\lambda_{a,a^{-1}}\lambda_{d,c}\lambda_{dc})u_{acd} \\ &\quad - (\lambda_{d,c}\lambda_{dc,a} + f(a)\lambda_{a,a^{-1}}\lambda_{c,d}\lambda_{cd,a^{-1}})u_{adc}. \end{aligned}$$

By (6) we obtain $S_f(c, d, a) = 0$.

Case 8). Let $\langle c, d \rangle$ be a dihedral group of order 8 and $c^2 = d^2 = 1$. Then $f(d) = \lambda_{d,d}$, $f(c) = \lambda_{c,c}$ and $S_f(c, d, a) = 2u_c u_d(u_a - u_a^f) = 0$. $\square$

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