

Iteration of some integral $T_{p,q}$

By ELIGIUSZ MIELOSZYK (Gdańsk)

An operational calculus $CO(L^0, L^1, S, T_q, s_q, Q)$ is a set $(L^0, L^1, S, T_q, s_q, Q)$, where L^1, L^0 are linear spaces. The linear operation $S : L^1 \rightarrow L^0$ called a derivative is a surjection. The set Q is a set of indices q for the linear operation $T_q : L^0 \rightarrow L^1$ and for the linear operation $s_q : L^1 \rightarrow \text{Ker } S$ such that

$$\begin{aligned} ST_q f &= f \quad \text{for } f \in L^0, \quad q \in Q, \\ T_q Sx &= x - s_q x \quad \text{for } x \in L^1, \quad q \in Q. \end{aligned}$$

The operation T_q is called an integral. The operation s_q is called a limit condition. (For the definition and properties of an operational calculus see for example [1–3].)

In this paper we present the formula for the iteration of an operation $T_{p,q}$ given by (6) and the application of this formula to solving the following problem

$$\begin{aligned} S_p^n x &:= (S + p \text{id})^n x = f, \\ s_{p,q}(S + p \text{id})^i x &= x_i^p \in \text{Ker } S_p \quad \text{for } i = 0, 1, \dots, n-1, \end{aligned}$$

where id is an identity operation and the operation $s_{p,q}$ is given by formula (7).

Let an operational calculus $CO(L^0, L^1, S, T_q, s_q, Q)$ be given, where $L^1 \subset L^0$, L^1, L^0 are commutative algebras with unity $\mathbf{1}$ over the field of real numbers and with the multiplication such that for $x, y \in L^1$

- (1) $S(x \cdot y) = (Sx) \cdot y + x \cdot (Sy),$
- (2) $s_q(x \cdot y) = (s_q x) \cdot (s_q y).$

Definition 1. ([5]) We will say that there exists an element $u := E_1^{T_{qp}}$ if and only if the element $E_1^{T_{qp}}$ is a solution of the abstract differential equation

$$(3) \quad Su = p \cdot u$$

with condition

$$(4) \quad s_q u = \mathbf{1},$$

where $u \in L^1$, $p \in L^0$ and $E_1^{T_{qp}} \in \text{Inv}$.

Theorem 1. (see [5]) *If there exists an element $E_1^{T_{qp}}$ then the three operations*

$$(5) \quad S_p u := Su + pu,$$

$$(6) \quad T_{p,q} f := \left[T_q \left(f \cdot E_1^{T_{qp}} \right) \right] \cdot E_1^{-T_{qp}}$$

$$(7) \quad s_{p,q} := (s_q u) \cdot E_1^{-T_{qp}}$$

satisfy the axioms of operational calculus, where $u \in L^1$, $f \in L^0$. The operation S_p is a derivative, the operation $T_{p,q}$ is an integral, the operation $s_{p,q}$ is a limit condition.

Definition 2. (see [5]) If there exists an element $E_1^{T_{qp}}$ then for the elements $x, y \in L^0$ we will define the multiplication $x \circ y$ by the formula

$$(8) \quad x \circ y := x \cdot y \cdot E_1^{T_{qp}}.$$

Corollary 1. (see [5]) *The multiplication $\circ$ satisfies condition (1) for the derivative S_p and condition (2) for the limit condition $s_{p,q}$. For the multiplication $\circ$ the unity $\mathbf{1}_\circ$ is defined by the formula*

$$\mathbf{1}_\circ = E_1^{-T_{qp}}.$$

Theorem 2. We have

$$(9) \quad T_{p,q}^n f = \left[T_q^n \left(f \cdot E_1^{T_{qp}} \right) \right] \cdot E_1^{-T_{qp}}, \quad \text{where } f \in L^0.$$

PROOF. We proceed by induction. The formula is true for $n = 1$. We must show that

$$(10) \quad T_{p,q}^{n+1} f = \left[T_q^{n+1} \left(f \cdot E_1^{T_{q^p}} \right) \right] \cdot E_1^{-T_{q^p}}.$$

From our inductive assumption and from the definition of the operation $T_{p,q}$ we will get

$$\begin{aligned} T_{p,q}^{n+1} f &= T_{p,q} T_{p,q}^n f = \\ &= \left[T_q \left((T_{p,q}^n f) \cdot E_1^{T_{q^p}} \right) \right] \cdot E_1^{-T_{q^p}} = \\ &= \left[T_q \left\{ \left(\left[T_q^n \left(f \cdot E_1^{T_{q^p}} \right) \right] \cdot E_1^{-T_{q^p}} \right) \cdot E_1^{T_{q^p}} \right\} \right] \cdot E_1^{-T_{q^p}} = \\ &= \left[T_q^{n+1} \left(f \cdot E_1^{T_{q^p}} \right) \right] \cdot E_1^{-T_{q^p}} \end{aligned}$$

□

Corollary 2.

$$T_{p,q}^n \mathbf{1}_o = [T_q^n \mathbf{1}] \cdot E_1^{-T_{q^p}}$$

Definition 3. (see [2,3]) Let

$$L^n := \{x \in L^{n-1} : Sx \in L^{n-1}\}, \quad n = 2, 3, \dots$$

Theorem 3. *The abstract differential equation*

$$(11) \quad (S + p \text{id})^n x = f$$

with conditions

$$(12) \quad s_{p,q} S_p^i x = x_i^p \in \text{Ker } S_p \quad \text{for } i = 0, 1, 2, \dots, n-1$$

where $x \in L^n$, $p \in L^{n-1}$, $f \in L^0$, $\text{id } x = x$ has only one solution given by the formula

$$(13) \quad x = x_0^p + T_{p,q} x_1^p + T_{p,q}^2 x_2^p + \dots + T_{p,q}^{n-1} x_{n-1}^p + T_{p,q}^n f.$$

PROOF. It is known from operational calculus that the abstract differential equation

$$S^n x = f, \quad f \in L^0, \quad x \in L^n$$

with conditions

$$s_q S^i x = x_i \in \text{Ker } S, \quad i = 0, 1, \dots, n-1$$

has only one solution x of the form

$$x = \sum_{i=0}^{n-1} T_q^i s_q S^i x + T_q^n f = \sum_{i=0}^{n-1} T_q^i x_i + T_q^n f. \quad (\text{see [2,3]})$$

Making use of the last fact it is easy to notice that formula (13) is true

□

Corollary 3. *If we introduce multiplication $\circ$ in L^0 then the solution (13) of the problem (11), (12) can be written in the form*

$$(14) \quad \begin{aligned} x = & x_0^p + x_1^p \circ T_{p,q} \mathbf{1}_\circ + x_2^p \circ T_{p,q}^2 \mathbf{1}_\circ + \cdots + \\ & + x_{n-1}^p \circ T_{p,q}^{n-1} \mathbf{1}_\circ + T_{p,q}^n f. \end{aligned}$$

Theorem 4. *The abstract differential equation (11) with conditions*

$$(15) \quad s_q S^i x = x_i \in \text{Ker } S \quad \text{for } i = 0, 1, \dots, n-1$$

has only one solution.

PROOF. The conditions (15) are equivalent to conditions (16), i.e. knowing conditions (15) we can define conditions (12) and conversely. Thus it follows from theorem 3 that the problem (11), (15) has only one solution.

□

Remark. For arbitrary n it is difficult to find the solution of the problem (11), (15). Therefore, as an example we will formulate a theorem which will give the form of the solution for $n = 3$.

Theorem 5. *The abstract differential equation*

$$(16) \quad (S + p \text{id})^3 x = f$$

with conditions

$$(17) \quad s_q S^i x = x_i, \quad i = 0, 1, 2$$

where $x \in L^3$, $p \in L^2$, $f \in L^0$, $x_i \in \text{Ker } S$ for $i = 0, 1, 2$ has only one solution given by the formula

$$(18) \quad \begin{aligned} x = & x_0 \cdot E_1^{-T_{q^p}} + T_{p,q} \left\{ (x_1 + x_0(s_q p)) \cdot E_1^{-T_{q^p}} \right\} + \\ & + T_{p,q}^2 \left\{ (x_2 + 2x_1 \cdot (s_q p) + x_0(s_q S p + (s_q p)^2)) \cdot E_1^{-T_{q^p}} \right\} + T_{p,q}^3 f. \end{aligned}$$

PROOF. Applying conditions (17) we can write

$$s_{p,q}x = x_0 \cdot E_1^{-T_{q^p}} = x_0^p,$$

$$s_{p,q}S_px = [x_1 + x_0 \cdot (s_qp)] \cdot E_1^{-T_{q^p}} = x_1^p,$$

$$s_{p,q}S_p^2x = [x_2 + 2(s_qp) \cdot x_1 + ((s_qp)^2 + s_qS_p) \cdot x_0] \cdot E_1^{-T_{q^p}} = x_2^p.$$

Substituting these conditions into formula (13) in theorem 3 for $n = 3$ we will get formula (18) as the solution x of the problem (16), (17). It follows from theorem 4 that it is the only solution of the problem (16), (17). $\square$

Example A. Let us consider an operational calculus with the derivative

$$S\{u(x_1, x_2, \dots, x_m)\} := \left\{ \sum_{i=1}^m b_i \frac{\partial u(x_1, x_2, \dots, x_m)}{\partial x_i} \right\}$$

the integral

$$\begin{aligned} T_{x_m^0} \{f(x_1, x_2, \dots, x_m)\} &:= \\ &:= \left\{ \frac{1}{b_m} \int_{x_m^0}^{x_m} f \left(x_1 - \frac{b_1}{b_m}(x_m - \tau), \dots, x_{m-1} - \frac{b_{m-1}}{b_m}(x_m - \tau), \tau \right) d\tau \right\}, \end{aligned}$$

and the limit condition

$$\begin{aligned} s_{x_m^0} \{u(x_1, x_2, \dots, x_m)\} &:= \\ &:= \left\{ u \left(x_1 - \frac{b_1}{b_m}(x_m - x_m^0), \dots, x_{m-1} - \frac{b_{m-1}}{b_m}(x_m - x_m^0), x_m^0 \right) \right\}, \end{aligned}$$

where $u \in L^1 := C^2(R^{m-1} \times \langle x_m^1, x_m^2 \rangle, R)$,

$f \in L^0 := C^1(R^{m-1} \times \langle x_m^1, x_m^2 \rangle, R)$,

$x_m^0 \in \langle x_m^1, x_m^2 \rangle$, $b_i \in R$ for $i = 1, 2, \dots, m$, $b_m \neq 0$ (see [4]).

For such a model of operational calculus operations S_p , T_{p,x_m^0} , s_{p,x_m^0} are defined in [5]. Following this paper we put

$$\begin{aligned} (19) \quad S_p \{u(x_1, x_2, \dots, x_m)\} &:= \left\{ \sum_{i=1}^m b_i \frac{\partial u(x_1, x_2, \dots, x_m)}{\partial x_i} + \right. \\ &\quad \left. + p(x_1, x_2, \dots, x_m)u(x_1, x_2, \dots, x_m) \right\}, \end{aligned}$$

$$(20) \quad T_{p,x_m^0} \left\{ f(x_1, x_2, \dots, x_m) \right\} :=$$

$$\begin{aligned}
& := \left\{ e^{-\frac{1}{b_m} \int_{x_m^0}^{x_m} p\left(x_1 - \frac{b_1}{b_m}(x_m - \tau), x_2 - \frac{b_2}{b_m}(x_m - \tau), \dots, x_{m-1} - \frac{b_{m-1}}{b_m}(x_m - \tau), \tau\right) d\tau} \right. \\
& \quad \cdot \frac{1}{b_m} \int_{x_m^0}^{x_m} f\left(x_1 - \frac{b_1}{b_m}(x_m - \tau), \dots, x_{m-1} - \frac{b_{m-1}}{b_m}(x_m - \tau), \tau\right) \cdot \\
& \quad \cdot e^{-\frac{1}{b_m} \int_{x_m^0}^{\tau} p\left(x_1 - \frac{b_1}{b_m}(x_m - \xi), \dots, x_{m-1} - \frac{b_{m-1}}{b_m}(x_m - \xi)\right) d\xi} d\tau \Bigg\}, \\
(21) \quad & s_{p, x_m^0} \left\{ u(x_1, x_2, \dots, x_m) \right\} := \\
& := \left\{ u\left(x_1 - \frac{b_1}{b_m}(x_m - x_m^0), x_2 - \frac{b_2}{b_m}(x_m - x_m^0), \dots \right. \right. \\
& \quad \left. \left. \dots, x_{m-1} - \frac{b_{m-1}}{b_m}(x_m - x_m^0), x_m^0\right) \cdot \right. \\
& \quad \left. \cdot e^{-\frac{1}{b_m} \int_{x_m^0}^{x_m} p\left(x_1 - \frac{b_1}{b_m}(x_m - \tau), \dots, x_{m-1} - \frac{b_{m-1}}{b_m}(x_m - \tau), \tau\right) d\tau} \right\},
\end{aligned}$$

where $u \in L^1$, $f, p \in L^0$.

In [6] there is shown a formula for the iteration of the integral $T_{x_m^0}$. It has the form

$$\begin{aligned}
(22) \quad & T_{x_m^0}^n \left\{ f(x_1, x_2, \dots, x_m) \right\} = \\
& = \left\{ \left(\frac{1}{b_m} \right)^n \int_{x_m^0}^{x_m} \frac{(x_m - \tau)^{n-1}}{(n-1)!} f\left(x_1 - \frac{b_1}{b_m}(x_m - \tau), \dots \right. \right. \\
& \quad \left. \left. \dots, x_{m-1} - \frac{b_{m-1}}{b_m}(x_m - \tau), \tau\right) d\tau \right\}.
\end{aligned}$$

In [6] it is also shown that for $c \in \text{Ker} \left(\sum_{i=1}^m b_i \frac{\partial}{\partial x_i} \right)$ the formula

$$(23) \quad T_{x_m^0}^n c = c \left(\left(\frac{1}{b_m} \right)^n \frac{(x_m - x_m^0)^n}{n!} \right)$$

is true. From the formulas (20), (22) and Theorem 2 we have

$$(24) \quad T_{p, x_m^0}^n \left\{ f(x_1, x_2, \dots, x_m) \right\} =$$

$$\begin{aligned}
&= \left(\frac{1}{b_m}\right)^n \left\{ \int_{x_m^0}^{x_m} \frac{(x_m - \tau)^{n-1}}{(n-1)!} f\left(x_1 - \frac{b_1}{b_m}(x_m - \tau), \dots \right. \right. \\
&\quad \left. \left. \dots, x_{m-1} - \frac{b_{m-1}}{b_m}(x_m - \tau), \tau\right) \cdot \right. \\
&\quad \left. \cdot e^{\frac{1}{b_m} \int_{x_m^0}^{\tau} p\left(x_1 - \frac{b_1}{b_m}(x_m - \xi), \dots, x_{m-1} - \frac{b_{m-1}}{b_m}(x_m - \xi), \xi\right) d\xi} d\tau \right\} \cdot \\
&\quad \cdot \left\{ e^{-\frac{1}{b_m} \int_{x_m^0}^{x_m} p\left(x_1 - \frac{b_1}{b_m}(x_m - \tau), \dots, x_{m-1} - \frac{b_{m-1}}{b_m}(x_m - \tau), \tau\right) d\tau} \right\},
\end{aligned}$$

where $f \in L^0$. It follows from (24) that for

$$c^p = c \left\{ e^{-\frac{1}{b_m} \int_{x_m^0}^{x_m} p\left(x_1 - \frac{b_1}{b_m}(x_m - \tau), \dots, x_{m-1} - \frac{b_{m-1}}{b_m}(x_m - \tau), \tau\right) d\tau} \right\},$$

where $c \in \text{Ker} \left(\sum_{i=1}^m b_i \frac{\partial}{\partial x_i} \right)$, i.e. for $c^p \in \text{Ker } S_p$ the formula

$$T_{p, x_m^0}^n c^p = c^p \left\{ \left(\frac{1}{b_m}\right)^n \frac{(x_m - x_m^0)^n}{n!} \right\}$$

is true.

Example B. It follows from theorem 3 and example A that the partial differential equation

$$\begin{aligned}
(25) \quad &\left(\sum_{i=1}^m b_i \frac{\partial}{\partial x_i} + p(x_1, x_2, \dots, x_m) \text{id} \right)^n \left\{ x(x_1, x_2, \dots, x_m) \right\} = \\
&= \left\{ f(x_1, x_2, \dots, x_m) \right\}
\end{aligned}$$

with conditions

$$\begin{aligned}
(26) \quad &s_{x_m^0} \left(\sum_{i=1}^m b_i \frac{\partial}{\partial x_i} + p(x_1, x_2, \dots, x_m) \text{id} \right)^i \left\{ x(x_1, x_2, \dots, x_m) \right\} = \\
&= \varphi_i \in \text{Ker} \left(\sum_{i=1}^m b_i \frac{\partial}{\partial x_i} \right) \quad \text{for } i = 0, 1, \dots, n-1,
\end{aligned}$$

where $x \in L^n$, $p \in L^{n-1}$, $f \in L^0$ (L^0 and L^1 are defined in example A while L^n is defined in definition 3), $x_m \in \langle x_m^1, x_m^2 \rangle$, $b_i \in R$ for $i = 1, 2, \dots, m$, $b_m \neq 0$ has only one solution given by

$$\begin{aligned} x &= \left\{ x(x_1, x_2, \dots, x_m) \right\} = \\ &= e^{-\frac{1}{b_m} \int_{x_m^0}^{x_m} p\left(x_1 - \frac{b_1}{b_m}(x_m - \tau), \dots, x_{m-1} - \frac{b_{m-1}}{b_m}(x_m - \tau), \tau\right) d\tau} \cdot \\ &\quad \left(\varphi_0 + \frac{1}{b_m} \varphi_1(x_m - x_m^0) + \left(\frac{1}{b_m}\right)^2 \varphi_2 \frac{(x_m - x_m^0)^2}{2!} + \dots \right. \\ &\quad \left. \left(\frac{1}{b_m}\right)^{n-1} \varphi_{n-1} \frac{(x_m - x_m^0)^{n-1}}{(n-1)!} \right) + T_{p, x_m^0}^n \left\{ f(x_1, x_2, \dots, x_m) \right\}, \end{aligned}$$

where $T_{p, x_m^0}^n \{f(x_1, x_2, \dots, x_m)\}$ is defined by the formula (24).

Example C. The partial differential equation (25) for $n = 3$ with conditions

$$\begin{aligned} \left\{ x(x_1, x_2, \dots, x_m^0) \right\} &= \left\{ \psi_0(x_1, x_2, \dots, x_{m-1}) \right\}, \\ \sum_{i=1}^m b_i \frac{\partial}{\partial x_i} \left\{ x(x_1, x_2, \dots, x_{m-1}, x_m^0) \right\} &= \left\{ \psi_1(x_1, x_2, \dots, x_{m-1}) \right\}, \\ \left(\sum_{i=1}^m b_i \frac{\partial}{\partial x_i} \right)^2 \left\{ x(x_1, x_2, \dots, x_{m-1}, x_m^0) \right\} &= \left\{ \psi_2(x_1, x_2, \dots, x_{m-1}) \right\}, \end{aligned}$$

where $\psi_i \in C^{4-i}(R^{m-1}, R)$ for $i = 0, 1, 2$ has on the basis of theorem 5 only one solution

$$\begin{aligned} x &= e^{-\frac{1}{b_m} \int_{x_m^0}^{x_m} p\left(x_1 - \frac{b_1}{b_m}(x_m - \tau), \dots, x_{m-1} - \frac{b_{m-1}}{b_m}(x_m - \tau), \tau\right) d\tau} \\ &\quad \left\{ \psi_0(v_1, v_2, \dots, v_{m-1}) + \frac{1}{b_m} \psi_1(v_1, v_2, \dots, v_{m-1}) + \right. \\ &\quad \left. + \psi_0(v_1, v_2, \dots, v_{m-1}) p(v_1, v_2, \dots, v_{m-1}, x_m^0)(x_m - x_m^0) + \right. \\ &\quad \left. + \left[\psi_2(v_1, v_2, \dots, v_{m-1}) + \right. \right. \\ &\quad \left. \left. + 2\psi_1(v_1, v_2, \dots, v_{m-1}) p(v_1, v_2, \dots, v_{m-1}, x_m^0) + \right. \right. \\ &\quad \left. \left. + \psi_0(v_1, v_2, \dots, v_{m-1}) \left(s_{x_m^0} \sum_{i=1}^m b_i \frac{\partial}{\partial x_i} \{p(x_1, x_2, \dots, x_m)\} \right) \right] \right\} \end{aligned}$$

$$+ \left(p(v_1, v_2, \dots, v_{m-1}, x_m^0) \right)^2 \Bigg] \left(\frac{1}{b_m} \right)^2 \frac{(x_m - x_m^0)^2}{2!} \Bigg\} + \\ + T_{p, x_m^0}^3 \left\{ f(x_1, x_2, \dots, x_m) \right\},$$

where $v_i = x_i - \frac{b_i}{b_m}(x_m - x_m^0)$ for $i = 1, 2, \dots, m-1$ and $T_{p, x_m^0}^3$ is defined by the formula (24) for $n = 3$.

Similarly, further examples for the application of the theorems formulated can be given, making use of other models of operational calculus (models with different derivatives S).

References

- [1] R. BITTNER, On certain axiomates for the operational calculus, *Bull. Acad. Polon. Sci. Cl. III*, **7** (1959), 1–9.
- [2] R. BITTNER, Algebraic and analytic properties of solution of abstract differential equations, *Rozprawy Matemat.* **41** (1964), 1–63.
- [3] R. BITTNER, Rachunek operatorów w przestrzeniach liniowych, *Warszawa*, 1974.
- [4] E. MIELOSZYK, Application of the operational calculus in solving partial difference equation, *Acta Mathematica Hungarica* **48** (1986), 118–130.
- [5] E. MIELOSZYK, Operational calculus in algebras, *Publicationes Mathematicae* **34** (1987), 137–143.
- [6] E. MIELOSZYK, Operation $T^k(x_m^0)$ and its application, *Zeszyty Naukowe PG, Matematyka* **15** (1991), 35–40.

ELIGIUSZ MIELOSZYK
TECHNICAL UNIVERSITY OF GDAŃSK
MAJAKOWSKIEGO 11/12
80-952. GDAŃSK
POLAND

(Received April 9, 1990)