

On an Nikolskii-type inequality

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I. Introduction and result

S. M. NIKOLSKII [7] proved in 1951 an inequality of the form

$$(1.1) \quad \|T_n(x)\|_{L^q[0,2\pi]} \leq C n^{\frac{1}{p}-\frac{1}{q}} \|T_n(x)\|_{L^p[0,2\pi]}$$

where C is a constant independent of $T_n(x)$ and n , $1 \leq p < q \leq \infty$ and $T_n(x)$ is a trigonometric polynomial of order at most n . This inequality has had many applications in approximation theory, for example to embedding problems (see e.g. [10]).

N. X. KY [2] introduced so-called $\{N, \lambda\}$ systems ($\lambda > 0$), that is orthogonal systems $\omega = \{\omega_n(x)\}_{n=0}^{\infty}$, $x \in [a, b]$, $-\infty \leq a, b \leq \infty$, such that for each $\phi_n \in \pi_n(\omega)$ the following inequality is valid:

$$(1.2) \quad \|\phi_n(x)\|_{L^q[a,b]} \leq C n^{\lambda|\frac{1}{p}-\frac{1}{q}|} \|\phi_n(x)\|_{L^p[a,b]} \quad 1 \leq p, q \leq \infty,$$

where $\pi_n(\omega)$ is the set of the form $\sum_{k=0}^n a_k \omega_k(x)$, $a_k \in \mathbf{R}$, $k=0, 1, 2, \dots, n$.

$n \in \mathbf{P} = \{0, 1, 2, \dots\}$ and C is a constant independent of $p_n(x)$ and n .

From previous literature we can see that the classical orthonormal systems are all of such type (see e.g. [1], [4], [5]). In this paper we give an inequality of type (1.2) for the General Hermite system, that is the system $\{p_n(x) \cdot e^{-x^2/2} |x|^{\alpha/2}\}$ ($\alpha > 0$), where $p_n(x)$ is the n -th orthonormal polynomial with respect to the weight $w(\alpha, x) = e^{-x^2/2} |x|^{\alpha/2}$ ($\alpha > 0$, $-\infty < x < \infty$). More precisely, we shall prove the following

Theorem. *Let $\alpha > 0$. For each $p_n \in \mathcal{P}$ ($n \in \mathbf{N}$) the following inequality is valid:*

$$(1.3) \quad \|p_n(x)w(\alpha, x)\|_{L^1(-\infty, \infty)} \leq C n^{1/2} \|p_n(x)w(\alpha, x)\|_{L^\infty(-\infty, \infty)}$$

Here (and throughout the paper) C is an absolute constant, which may be different at each occurrence. $\mathcal{P}_n$ is the set of algebraic polynomials of degree at most n . Moreover, the inequality (1.3) is sharp in the sense that there exists a sequence $\{\tilde{p}_n(x)\}_{n=0}^\infty$ of polynomials satisfying:

$$\|\tilde{p}_n(x)w(\alpha, x)\|_{L^1(-\infty, \infty)} \geq Cn^{1/2} \|\tilde{p}_n(x)w(\alpha, x)\|_{L^\infty(-\infty, \infty)}$$

II. Notations and lemmas

We shall use the following notations applied in [4].

$$(2.1) \quad L_{U(\alpha)}^p := \left\{ f; \|f(x)u(\alpha, x)\|_{L^p(0, \infty)} < \infty; \right. \\ \left. u(\alpha, x) = e^{-x/2}x^{\alpha/2} \right\}$$

$$(2.2) \quad L_{H(\alpha)}^p := \left\{ f; \|f(x)w(\alpha, x)\|_{L^p(-\infty, \infty)} < \infty; \right. \\ \left. w(\alpha, x) = e^{-x^2/2}|x|^{\alpha/2} \right\} \\ (\alpha > -1, 1 \leq p \leq \infty)$$

We denote the Laguerre polynomials and normalized Laguerre polynomials by

$$(2.3) \quad L_n^\alpha(x) = (n!)^{-1}x^{-\alpha}e^{+x} \left(\frac{d}{dx} \right)^n (x^{\alpha+n}e^{-x})$$

and by

$$(2.4) \quad \hat{L}_n(x) = (-1)^n \left(\frac{n!}{\Gamma(n + \alpha + 1)} \right)^{1/2} L_n^\alpha(x) \\ (x > 0, \alpha > -1, n \in \mathbf{P})$$

respectively.

The normalized General Hermite polynomials will be denoted by $\hat{H}_n^\alpha(x)$ ($\alpha > 1, x \in \mathbf{R}$) i.e.

$$(2.5) \quad \int_{-\infty}^{\infty} \hat{H}_n^\alpha(x) \hat{H}_m^\alpha(x) e^{-x^2} |x|^\alpha dx = \delta_{nm}$$

where δ_{nm} is the Kronecker symbol.

The Laguerre functions and General Hermite functions are denoted by

$$(2.6) \quad \mathcal{L}_n^\alpha(x) = e^{-x/2} x^{\alpha/2} \hat{L}_n^\alpha(x)$$

and by

$$(2.7) \quad \mathcal{H}_n^\alpha(x) = e^{-x^2/2} |x|^{\alpha/2} \hat{H}_n^\alpha(x) \\ (\alpha > -1, n \in \mathbf{P})$$

respectively.

Lemma 1. Let $V_{n,2n-1}^{H(\alpha)}(f; x)$ be the de La Vallée-Poussin mean of the General Hermite expansion of a function $f \in L_{H(\alpha)}^p (1 \leq p \leq \infty)$, then

$$(2.8) \quad V_{n,2n-1}^{H(\alpha)}(f; x) = \int_{-\infty}^{\infty} f(t) \sum_{k=0}^{2n-1} \Theta_{n,2n-1}(k) \hat{H}_k^\alpha(x) \hat{H}_k^\alpha(t) e^{-t^2} |t|^\alpha dt$$

$$\text{where } \Theta_{n,2n-1}(k) = \begin{cases} 1 & , k \leq n \\ \frac{2n-k}{n} & , n+1 \leq k \leq 2n-1 \end{cases}$$

PROOF. For $f \in L_{H(\alpha)}^p$, its Hermite-Fourier series is

$$(2.9) \quad \sum_{k=0}^{\infty} a_k \hat{H}_k^\alpha(x), \quad \text{where } a_k = \int_{-\infty}^{\infty} f(t) \hat{H}_k^\alpha(t) e^{-t^2} |t|^\alpha dt$$

If we put $S_n(f; x) = \sum_{k=0}^n a_k \hat{H}_k^\alpha(x)$, then

$$V_{n,2n-1}^{H(\alpha)}(f; x) = \frac{1}{n} \left(\sum_{k=0}^{2n-1} S_k(f; x) - \sum_{k=0}^{n-1} S_k(f; x) \right).$$

Using Abel transform, we have

$$(2.10) \quad \begin{aligned} \sum_{k=0}^{2n-1} S_k(f; x) &= \sum_{k=0}^{2n-1} \sum_{i=0}^k \int_{-\infty}^{\infty} f(t) \hat{H}_i^\alpha(t) \hat{H}_i^\alpha(x) e^{-t^2} |t|^\alpha dt \\ &= \sum_{k=0}^{2n-1} \int_{-\infty}^{\infty} f(t) \hat{H}_k^\alpha(t) \hat{H}_k^\alpha(x) e^{-t^2} |t|^\alpha dt \left(\sum_{i=k}^{2n-1} 1 \right) \\ &= \sum_{k=0}^{2n-1} (2n-k) \int_{-\infty}^{\infty} f(t) \hat{H}_k^\alpha(t) \hat{H}_k^\alpha(x) e^{-t^2} |t|^\alpha dt \end{aligned}$$

analogously,

$$(2.11) \quad \sum_{k=0}^{n-1} S_k(f; x) = \sum_{k=0}^{n-1} (n-k) \int_{-\infty}^{\infty} f(t) \hat{H}_k^\alpha(t) \hat{H}_k^\alpha(x) e^{-t^2} |t|^\alpha dt.$$

(2.10) and (2.11) lead to (2.8). q.e.d.

Denoting by $(C, 1)_n^{H(\alpha)}(f; x)$ the first Cesàro mean of (2.9) we have

$$(2.12) \quad (C, 1)_n^{H(\alpha)}(f; x) = \frac{1}{n} \sum_{k=0}^{n-1} \left(\sum_{j=0}^k a_j \hat{H}_j^\alpha(x) \right).$$

We can see that

$$(2.13) \quad (C, 1)_n^{H(\alpha)}(f; x) = \int_{-\infty}^{\infty} f(t) \left(\sum_{k=0}^n \left(1 - \frac{k}{n+1} \right) \hat{H}_k^\alpha(x) \hat{H}_k^\alpha(t) \right) e^{-t^2} |t|^\alpha dt$$

Lemma 2.

$$(2.14) \quad \hat{H}_{2n}^\alpha(x) = \hat{L}_n^{\frac{\alpha-1}{2}}(x^2)$$

$$(2.15) \quad \hat{H}_{2n+1}^\alpha(x) = \hat{L}_n^{\frac{\alpha+1}{2}}(x^2)x \quad (\alpha > -1, x \in \mathbf{R})$$

PROOF. Let $y = x^2$, then

$$\begin{aligned} \int_{-\infty}^{\infty} \hat{L}_n^{\frac{\alpha-1}{2}}(x^2) \hat{L}_m^{\frac{\alpha-1}{2}}(x^2) e^{-x^2} |x|^\alpha dx &= \int_0^{\infty} \hat{L}_n^{\frac{\alpha-1}{2}}(y) \hat{L}_m^{\frac{\alpha-1}{2}}(y) e^{-y} y^{\alpha/2} y^{-1/2} dy \\ &= \int_0^{\infty} \hat{L}_n^{\frac{\alpha-1}{2}}(y) \hat{L}_m^{\frac{\alpha-1}{2}}(y) e^{-y} y^{\frac{\alpha-1}{2}} dy = \delta_{nm} \end{aligned}$$

analogously,

$$\begin{aligned} \int_{-\infty}^{\infty} \hat{L}_n^{\frac{\alpha+1}{2}}(x^2) x \hat{L}_m^{\frac{\alpha+1}{2}}(x^2) x e^{-x^2} |x|^\alpha dx \\ = \int_0^{\infty} \hat{L}_n^{\frac{\alpha+1}{2}}(y) \hat{L}_m^{\frac{\alpha+1}{2}}(y) e^{-y} y^{\frac{\alpha+1}{2}} dy = \delta_n \end{aligned}$$

finally,

$$\int_{-\infty}^{\infty} \hat{L}_n^{\frac{\alpha-1}{2}}(x^2) \hat{L}_m^{\frac{\alpha+1}{2}}(x^2) x e^{-x^2} |x|^\alpha dx = 0 \quad \forall n, m \in \mathbb{P}$$

as the integrand is an odd function. q.e.d.

Lemma 3.

$$(2.16) \quad |\mathcal{H}_{2n}^\alpha(x)| \leq C \begin{cases} \nu^{\frac{\alpha-1}{2}} x^{\frac{2\alpha-1}{2}} & , 0 \leq x^2 \leq 1/\nu \\ \nu^{-1/4} & , 1/\nu < x^2 \leq \nu/2 \\ [\nu/x^2 (\nu^{1/3} + |x^2 - \nu|)]^{-1/4} & , \nu/2 < x^2 \leq 3\nu/2 \\ x^{1/2} e^{-\gamma x^2} & , 3\nu/2 < x^2 \end{cases}$$

$$(2.17) \quad |\mathcal{H}_{2n+1}^\alpha(x)| \leq C \begin{cases} \nu^{\frac{\alpha+1}{2}} x^{\frac{2\alpha-1}{2}} & , 0 \leq x^2 \leq 1/\nu \\ \nu^{-1/4} & , 1/\nu < x^2 \leq \nu/2 \\ [\nu/x^2 (\nu^{1/3} + |x^2 - \nu|)]^{-1/4} & , \nu/2 < x^2 \leq 3\nu/2 \\ x^{1/2} e^{-\gamma x^2} & , 3\nu/2 < x^2 \end{cases}$$

where C, γ are positive constants independent of n , $\nu = 4n + \alpha + 1$.

PROOF. Let $y = x^2$. Using lemma 2. we have

$$|\mathcal{H}_{2n}^\alpha(x)| = |\hat{H}_{2n}^\alpha(x) e^{-x^2/2} |x|^{\alpha/2}| = |\hat{L}_n^{\frac{\alpha-1}{2}}(x) e^{-y/2} y^{\alpha/4}|$$

using [3], (2.8) we get (2.16). The proof of (2.17) is analogous. q.e.d.

Corollary 1. If $\alpha > 1/2$ then

$$(2.18) \quad |\mathcal{H}_{2n}^\alpha(x)| \leq C \begin{cases} \nu^{-1/12} & , 0 \leq x^2 \leq 3\nu/2 \\ x^{1/2} e^{-\gamma x^2} & , 3\nu/2 < x^2 \end{cases}$$

$$(2.19) \quad |\mathcal{H}_{2n+1}^\alpha(x)| \leq C \begin{cases} \nu^{3/4} & , 0 \leq x^2 \leq 3\nu/2 \\ x^{1/2} e^{-\gamma x^2} & , 3\nu/2 < x^2 \end{cases}$$

where C, γ are positive constants, $\nu = 4n + \alpha + 1$.

Lemma 4.

$$(2.20) \quad \|\hat{H}_n^\alpha(x)\|_{L_{H(\alpha)}^p} \sim \begin{cases} n^{1/2p-1/4} & , 1 \leq p \leq 4 \\ n^{-(1/12+1/6p)} & , 4 < p \leq \infty \end{cases}$$

$$(2.21) \quad \|\hat{H}_{n+1}^\alpha(x) - \hat{H}_{n-1}^\alpha(x)\|_{L_{H(\alpha)}^p} \sim n^{1/2p-1/4} \quad (n \rightarrow \infty)$$

PROOF. Assume that n is even, i.e. $n = 2m$ ($m \in \mathbb{N}$) (the case when n is odd, leads to a slight modification only).

Let $y = x^2$. We have

$$\begin{aligned} \|\hat{H}_{2m}^\alpha(x)\|_{L_{H(\alpha)}^p} &= \|\hat{L}_m^{\frac{\alpha-1}{2}}(x^2)\|_{L_{H(\alpha)}^p} = \left(\int_{-\infty}^{\infty} |\hat{L}_m^{\frac{\alpha-1}{2}}(x^2) e^{-x^2/2} |x|^{\alpha/2}|^p dx \right)^{1/p} \\ &= \left(\int_0^{\infty} |\hat{L}_m^{\frac{\alpha-1}{2}}(y) e^{-y/2} y^{\alpha/4-1/2p}|^p dy \right)^{1/p} = \|\hat{L}_m^{\frac{\alpha-1}{2}}(y)\|_{L_{U(\alpha/2-1/p)}^p} \end{aligned}$$

Put $a = \alpha/2 - 1/p$, $a + b = (\alpha - 1)/2$, then $a + b > -1$ and $a > -2/p$ as $\alpha > -1$. From [4], (2.9) we get

$$\begin{aligned} \|\hat{H}_{2m}^\alpha(x)\|_{L_{H(\alpha)}^p} &= \|\hat{L}_m^{\frac{\alpha-1}{2}}(y)\|_{L_{U(\alpha/2-1/p)}^p} \sim \\ &\sim \begin{cases} m^{1/2p-1/4} & , 1 \leq p \leq 4 \\ m^{-(1/12-1/6p)} & , 4 < p \leq \infty. \end{cases} \sim \begin{cases} n^{1/2p-1/4} & , 1 \leq p \leq 4 \\ n^{-(1/12+1/6p)} & , 4 < p \leq \infty \end{cases} \end{aligned}$$

(2.21) is proved from (2.20) and the fact that $n^{-(1/12+1/6p)} < n^{1/2p-1/4}$ for $n \in \mathbb{N}$, $4 < p \leq \infty$. q.e.d.

Lemma 5.

$$(2.22) \quad \text{ess sup } \Lambda_n(t; \alpha, \alpha - 1/2, 1) \leq C \quad , \quad \alpha > -1/2$$

$$(2.23) \quad \Lambda_n(t; \alpha, \alpha - 1/2, 0) \leq C \quad , \quad \alpha \geq -1/2$$

Here $\Lambda_n(t; \alpha, \gamma, \delta)$ is the Lebesgue function, defined by

$$(2.24) \quad \Lambda_n(t; \alpha, \gamma, \delta) = (A_n^\delta)^{-1} \int_0^\infty \left| \sum_{k=0}^n A_{n-k}^\delta L_k^\alpha(x) L_k^\alpha(t) \right| (t/x)^{\frac{\alpha-\gamma}{2}} dx$$

where

$$(2.25) \quad A_n^\delta = \binom{n+\delta}{n}$$

PROOF. (2.23) was proved in ([3] p. 30–31). Putting $r = -1/4$ into [8], Corollary of theorem 1, we obtain

$$\|(C, 1)_n^\alpha\|_{[L_{U(\alpha-1/2)}^1]} \leq C .$$

However

$$\| (C, 1)_n^\alpha \|_{[L^1_{U(\alpha-1/2)}]} = \text{ess sup} \Lambda_n(t; \alpha, \alpha - 1/2, 1)$$

(c.f. [3], (4.19)) so we get (2.22). q.e.d.

III. The proof of the Theorem

Using the notation (2.2) we shall show the existence of a constant C such that

$$(3.1) \quad \| p_n(x) \|_{L^1_{H(\alpha)}} \leq C n^{1/2} \| p_n(x) \|_{L^\infty_{H(\alpha)}}$$

for every $p_n \in \mathcal{P}_n$, and this inequality is sharp. Let $p_n \in \mathcal{P}_n$ be arbitrary, then

$$\begin{aligned} \| p_n(x) \|_{L^1_{H(\alpha)}} &= \| V_{n, 2n-1}^{H(\alpha)}(p_n; x) \|_{L^1_{H(\alpha)}} \\ &= \| 2(C, 1)_{2n-1}^{H(\alpha)}(p_n; x) - (C, 1)_{n-1}^{H(\alpha)}(p_n; x) \|_{L^1_{H(\alpha)}} \\ &\leq 2 \| (C, 1)_{2n-1}^{H(\alpha)}(p_n; x) \|_{L^1_{H(\alpha)}} + \| (C, 1)_{n-1}^{H(\alpha)}(p_n; x) \|_{L^1_{H(\alpha)}} \\ &\leq \{2I(2n-1) + I(n-1)\} \| P_n(x) \|_{L^\infty_{H(\alpha)}} \end{aligned}$$

where

$$\begin{aligned} I(n) &= \int_{-\infty}^{\infty} \left\| \sum_{k=0}^n \left(1 - \frac{k}{n+1}\right) \hat{H}_k^\alpha(t) \hat{H}_k^\alpha() \right\|_{L^1_{H(\alpha)}} e^{-t^2/2} |t|^{\alpha/2} dt \\ &= \int_{-\infty}^{\sqrt{3\nu/2}} + \int_{-\sqrt{3\nu/2}}^{\sqrt{3\nu/2}} + \int_{\sqrt{3\nu/2}}^{\infty} =: I_1(n) + I_2(n) + I_3(n). \\ I_3(n) &\leq \sum_{k=0}^n \left(1 - \frac{k}{n+1}\right) \int_{\sqrt{3\nu/2}}^{\infty} \| \hat{H}_k^\alpha(x) \|_{L^1_{H(\alpha)}} |\hat{H}_k^\alpha(t)| e^{-t^2/2} t^{\alpha/2} dt \\ &\leq \sum_{k=0}^n \| H_k^\alpha(x) \|_{L^1_{H(\alpha)}} \int_{\sqrt{3\nu/2}}^{\infty} |\hat{H}_k^\alpha(t)| e^{-t^2/2} t^{\alpha/2} dt \end{aligned}$$

Hence, using lemma 3. and lemma 4. we obtain

$$\begin{aligned}
 I_3(n) &\leq C \sum_{k=0}^n k^{1/4} \int_{\sqrt{3\nu/2}}^{\infty} x^{1/2} e^{-\gamma x^2} dx \\
 (3.2) \quad &\leq C n^{5/4} \int_{\sqrt{3\nu/2}}^{\infty} y^2 e^{-\gamma \nu^{3/4} y} dy \\
 &\leq C n^{5/4} e^{-\gamma \nu} n^{-1/4} \leq C n e^{\gamma n} \leq C
 \end{aligned}$$

Analogously, we can see that

$$(3.3) \quad I_1(n) \leq C$$

Now, let us prove that $I_2(n) \leq C n^{1/2}$ Let $n = 2m + 1$ (the case $n = 2m$ can be dealt with in a similar way with slight modification)

$$\begin{aligned}
 &\sum_{k=0}^n \left(1 - \frac{k}{n+1}\right) \hat{H}_k^\alpha(x) \hat{H}_k^\alpha(t) = \sum_{k=0}^{2m+1} \left(1 - \frac{k}{2m+2}\right) \hat{H}_k^\alpha(x) \hat{H}_k^\alpha(t) \\
 &= \sum_{j=0}^m \left(1 - \frac{2j}{2m+2}\right) \hat{H}_{2j}^\alpha(x) \hat{H}_{2j}^\alpha(t) + \sum_{j=0}^m \left(1 - \frac{2j+1}{2m+2}\right) \hat{H}_{2j+1}^\alpha(x) \hat{H}_{2j+1}^\alpha(t) \\
 &= \sum_{j=0}^m \left(1 - \frac{j}{m+1}\right) \hat{L}_j^{\frac{\alpha-1}{2}}(x^2) \hat{L}_j^{\frac{\alpha-1}{2}}(t^2) \\
 &\quad + \sum_{j=0}^m \left(1 - \frac{j}{m+1}\right) \hat{L}_j^{\frac{\alpha+1}{2}}(x^2) \hat{L}_j^{\frac{\alpha+1}{2}}(t^2) x t \\
 &\quad - \frac{1}{(2m+2)} \sum_{j=0}^m \hat{L}_j^{\frac{\alpha+1}{2}}(x^2) \hat{L}_j^{\frac{\alpha+1}{2}}(t^2) x t =: A_1 + A_2 + A_3
 \end{aligned}$$

$$\begin{aligned}
 I_2(n) &= \int_{-\sqrt{3\nu/2}}^{\sqrt{3\nu/2}} \int_{-\infty}^{\infty} \left| \sum_{k=0}^{2m+1} \left(1 - \frac{k}{2m+2}\right) \hat{H}_k^\alpha(x) \hat{H}_k^\alpha(t) \right| e^{-\frac{x^2+t^2}{2}} |x t|^{\alpha/2} dx dt \\
 &\leq 4 \sum_{j=1}^3 \int_0^{\sqrt{3\nu/2}} \int_0^{\infty} |A_j| e^{-\frac{x^2+t^2}{2}} (x t)^{\alpha/2} dx dt
 \end{aligned}$$

Putting $x = \xi^2$, $t = \tau^2$, we get

$$\begin{aligned}
 I_2(n) &= C \int_0^{3\nu/2} \int_0^\infty \left| \sum_{j=0}^m \left(1 - \frac{j}{m+1}\right) \hat{L}_j^{\frac{\alpha-1}{2}}(\xi) \hat{L}_j^{\frac{\alpha-1}{2}}(\tau) \right| e^{-\frac{\xi+\tau}{2}} (\xi\tau)^{\alpha/4-1/2} d\xi d\tau \\
 &\quad + C \int_0^{3\nu/2} \int_0^\infty \left| \sum_{j=0}^m \left(1 - \frac{j}{m+1}\right) \hat{L}_j^{\frac{\alpha+1}{2}}(\xi) \hat{L}_j^{\frac{\alpha+1}{2}}(\tau) \right| e^{-\frac{\xi+\tau}{2}} (\xi\tau)^{\alpha/4} d\xi d\tau \\
 &\quad + C \int_0^{3\nu/2} \int_0^\infty \left| \frac{1}{(2m+2)} \sum_{j=0}^m \hat{L}_j^{\frac{\alpha+1}{2}}(\xi) \hat{L}_j^{\frac{\alpha+1}{2}}(\tau) \right| e^{-\frac{\xi+\tau}{2}} (\xi\tau)^{\alpha/4} d\xi d\tau \\
 &= C \int_0^{3\nu/2} \int_0^\infty \left| \sum_{j=0}^m \left(1 - \frac{j}{m+1}\right) \mathcal{L}_j^{\frac{\alpha-1}{2}}(\xi) \mathcal{L}_j^{\frac{\alpha-1}{2}}(\tau) \right| (\xi\tau)^{-1/4} d\xi d\tau \\
 &\quad + C \int_0^{3\nu/2} \int_0^\infty \left| \sum_{j=0}^m \left(1 - \frac{j}{m+1}\right) \mathcal{L}_j^{\frac{\alpha+1}{2}}(\xi) \mathcal{L}_j^{\frac{\alpha+1}{2}}(\tau) \right| (\xi\tau)^{-1/4} d\xi d\tau \\
 &\quad + \frac{C}{(2m+2)} \int_0^{3\nu/2} \int_0^\infty \left| \sum_{j=0}^m \mathcal{L}_j^{\frac{\alpha+1}{2}}(\xi) \mathcal{L}_j^{\frac{\alpha+1}{2}}(\tau) \right| (\xi\tau)^{-1/4} d\xi d\tau \\
 &= C \int_0^{3\nu/2} \Lambda_m(\tau; \frac{\alpha-1}{2}, \frac{\alpha-2}{2}, 1) \tau^{-1/2} d\tau \\
 &\quad + C \int_0^{3\nu/2} \Lambda_m(\tau; \frac{\alpha+1}{2}, \frac{\alpha}{2}, 1) \tau^{-1/2} d\tau \\
 &\quad + \frac{C}{(2m+2)} \int_0^{3\nu/2} \Lambda_m(\tau; \frac{\alpha+1}{2}, \frac{\alpha}{2}, 0) \tau^{-1/2} d\tau
 \end{aligned}$$

Hence (2.22) and (2.23) of lemma 5. yield

$$\begin{aligned}
 (3.4) \quad I_2(n) &\leq C \int_0^{3\nu/2} \tau^{-1/2} d\tau + C \frac{m^{1/6}}{2m+2} \int_0^{3\nu/2} 1 d\tau \\
 &\leq C\nu^{1/2} \leq Cn^{1/2}
 \end{aligned}$$

(3.2), (3.3) and (3.4) lead to (3.1).

Finally, we show that (3.1) is sharp. We can choose the test polynomials as follows:

$$(3.5) \quad \tilde{p}_n(x) := \left\{ \hat{H}_n^\alpha(x) - \hat{H}_{n-2}^\alpha(x) \right\} \quad (x \in \mathbf{R}, n \geq 2)$$

Then, for $1 \leq p \leq \infty$, by (2.21)

$$\| \tilde{p}_n(x) \|_{L^p_{H(\alpha)}} = \| \hat{H}_n^\alpha(x) - \hat{H}_{n-2}^\alpha(x) \|_{L^p_{H(\alpha)}} \sim n^{1/2p-1/4}$$

Consequently,

$$\frac{\| \tilde{p}_n(x) \|_{L^1_{H(\alpha)}}}{\| \tilde{p}_n(x) \|_{L^\infty_{H(\alpha)}}} \sim n^{1/2}$$

Thus there exists a constant $C > 0$, such that for each $n \in \mathbf{N}$

$$\| \tilde{P}_n(x) \|_{L^1_{H(\alpha)}} \geq Cn^{1/2} \| \tilde{P}_n(x) \|_{L^\infty_{H(\alpha)}}$$

This completes the proof of the Theorem. q.e.d.

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