

The companions of inner mapping groups

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Abstract. The objective of this paper is to investigate the companions of inner mapping groups of some special classes of Moufang loops.

It is shown that

- (1) the set of companions of all the inner mappings of a pE loop G is G for $p \neq 3$ and is N , the nucleus, for $p = 3$.
- (2) the set of companions of all the inner mappings of an F loop G which is generated by three elements is NG^3G' .

Introduction

For what loop G is it true that every loop isotopic to G is isomorphic to G ? This question is of considerable geometric significance, particularly in relation to 3-nets. R. H. BRUCK proved in [1, p.64, Theorem 2.3] that a necessary and sufficient condition that every loop isotopic to a Moufang loop G be isomorphic to G is that every element of G be a companion of at least one pseudo-automorphism of G . Since every inner mapping of G is a pseudo-automorphism of G , thus this brings us to the study of the companions of inner mappings of some special classes of Moufang loops.

Definitions

An F loop is a Moufang loop such that if H is a subloop generated by any three elements x, y, z , then the associator $(x, y, z) \in Z(H)$, the centre of H .

A pE loop G is a Moufang loop such that G/N is commutative of exponent p , where N is the nucleus of G and p is a prime.

$I(G)$, the inner mapping group of the loop G is defined as $\langle R(x, y), L(x, y), T(x) \mid x, y \in G \rangle$ where $R(x, y) = R(x)R(y)R(xy)^{-1}$, $L(x, y) = L(x)L(y)L(yx)^{-1}$, $T(x) = R(x)L(x)^{-1}$.

Define $T(G) = \langle T(x) \mid x \in G \rangle$.

A permutation S of a loop G is called a pseudo-automorphism of G provided there exists at least one element c of G , called a companion of S , such that

$$(xS) \cdot (yS \cdot c) = (xy)S \cdot c$$

for all x, y in G . If c is a companion of S , then cN is obviously the set of all companions of S . It is known that every element of $I(G)$ is a pseudo-automorphism. Define $C[I(G)]$ and $C[T(G)]$ as the set of companions of $I(G)$ and $T(G)$ respectively.

G_a , the associator subloop of G , is generated by all the associators (x, y, z) where $xy \cdot z = (x \cdot yz)(x, y, z)$.

G_c , the commutator subloop of G , is generated by all the commutators (x, y) where $xy = yx \cdot (x, y)$.

G' , the associator-commutator subloop of G , is generated by G_a and G_c .

Facts

Let G be a Moufang loop.

- F1.* If $G = \langle x, y, z \rangle$ is an F loop, then $G_a = \langle (x, y, z) \rangle \subset Z$.
- F2.* $S \in I(G) \Rightarrow S$ is a pseudo-automorphism of G [1, p.117, Lemma 3.2].
- F3.* A companion of $T(x)$ is x^{-3} and a companion of $R(x, y) = L(x^{-1}, y^{-1})$ is (x, y) . [1, p.113, Lemma 2.2].
- F4.* If θ, ψ are pseudo-automorphisms of G with companions a, b respectively, then $\theta\psi$ has companion $(a\psi) \cdot b$ and θ^{-1} has companion $(a\theta^{-1})^{-1}$. [3, p.62, 2(iii)].
- F5.* If $\theta \in I(G)$, then $\theta = T(g)R(x_1, y_1) \dots R(x_n, y_n)$ where $g, x_i, y_i \in G$. [2, p.322, Theorem 10A].
- F6.* $C[T(G)] = NG^3$. [3, p.64, Theorem 2.2].
- F7.* If G is an F loop, then $gR(x, y) = g(g, x, y)$, for $g, x, y \in G$. [5, p. 294, Lemma 1].
- F8.* A pE loop is an F loop. [1, p.125, Lemma 5.5 (ii)].

Bruck's Lemma. *Let G be a Moufang loop. Then G satisfies all or none of the following identities:*

- (i) $((x, y, z), x) = 1$ (ii) $(x, y, (y, z)) = 1$
- (iii) $(x, y, z)^{-1} = (x^{-1}, y, z)$ (iv) $(x, y, z)^{-1} = (x^{-1}, y^{-1}, z^{-1})$
- (v) $(x, y, z) = (x, zy, z)$ (vi) $(x, y, z) = (x, xy, z)$
- (vii) $(x, y, z) = (x, z, y^{-1})$

When these identities hold, then the associator (x, y, z) lies in the centre of the subloop generated by x, y, z ; and the following identities hold for all integers n :

- (viii) $(x, y, z) = (y, z, x) = (y, x, z)^{-1}$
- (ix) $(x^n, y, z) = (x, y, z)^n$
- (x) $(xy, z) = (x, z) ((x, z), y) (y, z) (x, y, z)^3$

PROOF. [1, p.125, Lemma 5.5].

Remark. A Moufang loop G is an F loop if and only if G satisfies Bruck's Lemma. [1, p.125, Lemma 5.5].

Theorem 1. *If G is a pE loop with nucleus N , then $G = NG^3$ for $p \neq 3$ and $C[I(G)] = G$.*

PROOF. The fact that G/N is commutative of exponent p implies $G_c \subset N$ and $x^p \in N$ for all $x \in G$. As $(p, 3) = 1$, so $p = 3m \pm 1$. Thus, $x^p = x^{3m \pm 1} \in N$. Then, $x^{\pm 1} = x^p x^{-3m} \in NG^3$. Thus, $G = NG^3$.

As $T(G) \subset I(G)$, therefore $C[T(G)] \subset C[I(G)]$. By F6, $C[T(G)] = NG^3 = G$, so $G \subset C[I(G)]$. Obviously, $C[I(G)] \subset G$, thus $G = C[I(G)]$.

Remark. Since for $p \neq 3$, every element of the pE loop G is a companion of some pseudo-automorphism of G , every isotope of G is isomorphic to G [1, p.115, Theorem 2.3].

Theorem 2. *If G is a $3E$ loop with nucleus N , then $N = NG^3$ and $C[I(G)] = N$.*

PROOF. G is a $3E$ loop implies $x^3 \in N$ and thus $G^3 \subset N$. Then $NG^3 = N$. Let $\theta \in I(G)$, then by F5, $\theta = T(g)R(x_1, y_1) \dots R(x_n, y_n)$ where $g, x_i, y_i \in G$ for $i = 1, 2, \dots, n$.

A companion of

$$\begin{aligned}
 & T(g)R(x_1, y_1) = \\
 & = g^{-3}R(x_1, y_1) \cdot (x_1, y_1) = && \text{by } F4 \\
 & = g^{-3}(g^{-3}, x_1, y_1) \cdot (x_1, y_1) = && \text{by } F7
 \end{aligned}$$

$$= n_1 \in N \text{ as } G_c \subset N \text{ and } x^3 \in N$$

Then, a companion of

$$\begin{aligned} T(g)R(x_1, y_1)R(x_2, y_2) &= \\ = n_1 R(x_2, y_2) \cdot (x_2, y_2) &= \text{by } F4 \\ = n_1(n_1, x_2, y_2) \cdot (x_2, y_2) &= \text{by } F7 \\ = n_2 \in N \end{aligned}$$

Thus, we can deduce that, $C[I(G)] \subset N$. But, $N = NG^3 = C[T(G)] \subset C[I(G)]$. Therefore, $N = C[I(G)]$.

Remark. Commutative Moufang loops are $3E$ loops. Since there exist nonassociative commutative Moufang loops, so an isotope of a $3E$ loop G is not necessarily isomorphic to G by [1, p.58 (ix)]. This distinguishes $3E$ loops from other pE loops.

Theorem 3. If $G = \langle x, y, z \rangle$ is an F loop, then $NG^3G' = C[I(G)]$.

PROOF. By $F1$, $G_a \subset Z$. Then clearly $NG^3G_c = NG^3G'$. By $F5$, $\theta \in I(G) \Rightarrow \theta = T(g)R(g_1, h_1) \dots R(g_n, h_n)$; $g, h_i, g_i \in G$, $i = 1, 2, \dots, n$. The companions of

$$\begin{aligned} T(g)R(g_1, h_1) &= \\ = g^{-3}R(g_1, h_1) \cdot (g_1, h_1)N &= \text{by } F4 \\ = g^{-3}(g^{-3}, g_1, h_1) \cdot (g_1, h_1)N &= \text{by } F7 \\ = g^{-3}(g_1, h_1)N &= \text{as } G_a \subset Z \subset N \end{aligned}$$

The companions of

$$\begin{aligned} T(g)R(g_1, h_1)R(g_2, h_2) &= \\ = [g^{-3}(g_1, h_1)]R(g_2, h_2) \cdot (g_2, h_2)N &= \text{by } F4 \\ = [g^{-3}(g_1, h_1)](g^{-3}(g_1, h_1), g_2, h_2)(g_2, h_2)N &= \\ = (g^{-3}(g_1, h_1))(g_2, h_2)N &= \\ = g^{-3}[(g_1, h_1)(g_2, h_2)]N &= \text{as } G_a \subset Z \subset N \end{aligned}$$

Similarly, the companions of

$$\begin{aligned} T(g)R(g_1, h_1) \dots R(g_n, h_n) &= g^{-3}[(g_1, h_1) \dots (g_n, h_n)]N \\ &\subset G^3G_cN = NG^3G_c = NG^3G' \end{aligned}$$

$$\therefore C[I(G)] \subset NG^3G'.$$

Conversely, let $g \in NG^3$, $c \in G_c$. Then g is a companion of some $\theta \in T(G)$ by $F6$. Let $c = c_1c_2 \dots c_n$ be associated in some way, where $c_i = (x_i, y_i)$, $x_i, y_i \in G$, $i = 1, \dots, n$. By $F4$, we see that gc is a companion of

$$\theta R(x_1, y_1)R(x_2, y_2) \dots R(x_n, y_n)$$

Therefore $NG^3G_c \subset C[I(G)]$

$$\therefore NG^3G_c = C[I(G)].$$

Remark. We do not know whether this result still holds for an F loop with more than three generators.

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(Received May 26, 1990)