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Sylow subgroups of soluble linear groups

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0. Introduction

In this paper we examine some questions concerning the restrictions for the Sylow sugroups of soluble linear groups. This problem has been studied by many authors, mainly by B. HUPPERT [3], J.D. DIXON [2], D.L. WINTER [6], T.R. WOLF [7]. Let F be a field, p a prime, G a finite p-soluble completely reducible subgroup of the general linear group $GL(n, F)$ and let $\alpha_p(G)$ denote the exponent of the order of a Sylow p-subgroup of $G/O_p(G)$; some results of the mentioned authors solve the problem of finding functions $\alpha_p(x)$ (x a real number), for which $\alpha_p(G) \leq \alpha_p(n)$. In such a problem the functions

$$\lambda_p(x) = \sum_{i=1}^{\infty} \left[\frac{x}{p^i}\right] \quad \text{and} \quad \beta_p(x) = \sum_{i=0}^{\infty} \left[\frac{x}{(p-1)p^i}\right]$$

often occur ($[y]$ denotes the greatest integer $\leq y$, y a real number). The results of DIXON [2] concern the soluble linear groups and improve those of HUPPERT [3], except when the field has two elements; the results of WINTER [6] generalize to p-soluble groups those of DIXON; the results of WOLF [7] concern essentially the case in which the field has order p and, if $p = 2$, improve either the results of DIXON or those of HUPPERT. All these results can be summarized as follows:

Let p be a given prime, and let G be a finite p-soluble completely reducible subgroup of the general linear group $GL(n, F)$ over a field F. Then $\alpha_p(G) \leq \alpha_p(n)$, where:

$$\alpha_p(n) = \begin{cases} \beta_p(n) & \text{if } p \text{ is a Fermat prime} \\ \lambda_p(n) & \text{if } p \text{ is odd and not a Fermat prime} \\ \beta_2(2n/3) & \text{if } p = 2 \end{cases}$$

Moreover, if $F = GF(2)$, *then* $\alpha_2(n) = \lambda_2(n)$.

The values for $\alpha_p(n)$, when $p \neq 2$, are given by DIXON [2]; the value $\beta_2(2n/3)$ has been obtained by I.M. ISAACS ([5], p.257) for a field with characteristic zero and later used by WOLF [7] for a field with characteristic different from zero.
The limiting values, as the authors themselves notice, are best possible for each value of n, that is: for each integer $n \geq 1$ and for each prime p, there exist a field F and a finite soluble completely reducible subgroup G of $GL(n, F)$, such that $\alpha_p(G) \leq \alpha_p(n)$.

In this paper we find some sufficient conditions on the base field, which allow the attainment of the above-mentioned limiting values, and we prove that, for soluble linear groups, these conditions are necessary as well. Moreover we prove that, when such conditions fail, we can provide a better estimate for $\alpha_p(n)$; such estimate is best possible, because the limiting values are attained on a prime field.

1. The limiting values

In this section we prove that the limiting value $\beta_2(2n/3)$ is attained on a field which contains primitive cubics roots of unity and that, if p is a Fermat prime, to attain the limiting value $\beta_p(n)$ it is sufficient that the field has characteristic different from 2 and contains solutions of the equation $x^2+y^2 = -1$ (in particular, char$F$=q prime$\neq$2).

We use the following notation.
Let F be a field, char$F \neq 2$, and let p be an odd prime. The matrix group of degree p on F generated by the permutation matrix of a p-cycle together with all diagonal matrices with diagonal entries ± 1 will be denoted by $M(p, F)$.
Let p and q be primes. $N(p, q)$ will denote the normalizer of the Singer cycle of $GL(p, q)$. If F is a field, the subgroup of $GL(2, F)$:

$$\langle \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \rangle \simeq S_3$$

will be denoted by $S(F)$.

Lemma 1.1. *Let* F *be a field,* p *a prime,* d *and* n *integers such that* $1 \leq d < n$. *Put* $m = [n/d]$ *and* $r = n - md$, *let* G_0 *and* G_1 *be finite soluble completely reducible subgroups of* $GL(d, F)$ *and* $GL(s, F)$ *respectively, with* $s \leq r$, *and* G_0 *has no trivial component. Then there exists a finite*

soluble completely reducible subgroup G of $GL(n,F)$ such that $\alpha_p(G) = m\alpha_p(G_0) + \lambda_p(m) + \alpha_p(G_1)$.

PROOF. Without loss of generality, we can assume $n = md$. For, otherwise, if $\bar{G}$ is a soluble completely reducible subgroup of $GL(md,F)$, with $\alpha_p(\bar{G}) = m\alpha_p(G_0)+\lambda_p(m)$, it is enough to consider the $F(\bar{G}\times G_1)$-module direct sum of a faithful completely reducible $F\bar{G}$-module of degree md and a faithful completely reducible FG_1-module of degree r, which has a faithful completely reducible direct summand of degree s (the possible other one being trivial). Let then $n = md$ and let W be a faithful completely reducible FG_0-module of degree d. Let V be the direct sum of m-copies of W and H a Sylow p-subgroup of the symmetric group S_m. If G is the wreath product of G_0 by H, V is obviously a faithful completely reducible FG-module and $\alpha_p(G) = m\alpha_p(G_0) + \log_p|H| = m\alpha_p(G_0) + \lambda_p(m)$.

Proposition 1.2. *Let F be a field containing a primitive cubic root of unity. Then, for each integer $n \geq 1$, there exists a soluble completely reducible subgroup G of $GL(n,F)$ such that $\alpha_2(G) = \beta_2(2n/3)$.*

PROOF. By Lemma 1.1 it is enough to find a soluble irreducible subgroup G_0 of $GL(3,F)$, such that $\alpha_2(G_0) = 3$; after that, if $\mathrm{rem}(n,3) = 2$, we can take $G_1 = S(F)$. Let then ε be a primitive cubic root of unity and let G_0 be the subgroup of $GL(3,F)$:

$$G_0 = \Big\langle 1/\sqrt{-3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \varepsilon & \varepsilon^{-1} \\ 1 & \varepsilon^{-1} & \varepsilon \end{bmatrix},\ 1/\sqrt{-3}\begin{bmatrix} 1 & \varepsilon & \varepsilon \\ \varepsilon^{-1} & \varepsilon & \varepsilon^{-1} \\ \varepsilon^{-1} & \varepsilon^{-1} & \varepsilon \end{bmatrix}\Big\rangle$$
$$\ltimes\Big\langle \begin{bmatrix} 1 & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon^{-1} \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\Big\rangle.$$

It is immediate to verify that $G_0 \simeq Q_8 \ltimes P$, with P extraspecial of order 3^3 and exponent 3; thus $\alpha_2(G_0) = 3$.

Proposition 1.3. *Let p be a Fermat prime and let F be a field, $\mathrm{char}F \neq 2$, which contains solutions of the equation $x^2 + y^2 = -1$. Then, for each integer $n \geq 1$, there exists a soluble completely reducible subgroup G of $GL(n,F)$ such that $\alpha_p(G) = \beta_p(n)$.*

PROOF. Let $p = 2^{2^m}+1$, $m \geq 0$, and let N be the extraspecial group of order $2^{2^{m+1}+1}$ corresponding to the quadratic form of non-maximal Witt index. N possesses an automorphism σ of order p which acts irreducibly on $N/Z(N)$ (see [4], p.357); let $G_0 = \langle\sigma\rangle \ltimes N$. By Lemma 1.1 it is enough to prove that there exists a faithful irreducible FG_0-module of degree $p-1$. Since F contains solutions of the equation $x^2 + y^2 = -1$ and $\mathrm{char}F \neq 2$,

there exists a faithful absolutely irreducible FN-module W of degree $p-1$ and W is unique up to isomorphisms; we have then that the induced FG_0-module W^{G_0} is reducible (see [1], 45.5) and that, if V is a faithful irreducible FG_0-module, an irreducible N-submodule of V is isomorphic to W; it follows that $p-1$ divides $\dim_F V$. On the other hand we have that $\dim_F V$ divides $|G_0 : N| \dim_F W = p(p-1)$ (see [1], 53.17); it follows, being $V \not\simeq W^{G_0}$, that the degree of V is $p-1$.

The following result is well known.

Proposition 1.4. *Let p be a given prime. Then, for every field F and for each integer $n \geq 1$, there exists a soluble completely reducible subgroup G of $GL(n, F)$ such that $\alpha_p(G) = \lambda_p(n)$.*

Proof. It is enough to find a soluble irreducible subgroup G_0 of $GL(p, F)$ with $\alpha_p(G_0) = 1$ (Lemma 1.1). If $p = 2$ we can assume $G_0 = S(F)$ or, if $\mathrm{char}F = q \neq 0$, $G_0 = N(2, q)$. If $p \neq 2$, we can assume $G_0 = M(p, F)$, if $\mathrm{char}F \neq 2$, or, if $\mathrm{char}F = q \neq 0$, $G_0 = N(p, q)$.

2. Inequalities

In this section we provide some inequalities, which will be used repeatedly.

2.1. *Let t and r be positive integers, with $r > 1$. Then:*

$$\beta_2(2t/3) + \lambda_2(r) \leq \lambda_2(tr)$$

Proof. If $t \leq 2$, the statement is obvious. Let then be $t \geq 3$. We have:

$$\beta_2(2t/3) + \lambda_2(r) = \lambda_2(4t/3) + \lambda_2(r) \leq \lambda_2(4t/3 + r) \leq \lambda_2(tr).$$

2.2. *Let t and r be positive integers, with $r > 1$. Then:*

$$\beta_p(t) + \lambda_p(r) \leq \lambda_p(tr)$$

for each odd prime p.

Proof. If $t < p-1$ or $r < p$, the statement is obvious. Let then be $t \geq p-1$ and $r \geq p$ and choose the integers j and ℓ so that

$$(p-1)p^j \leq t < (p-1)p^{j+1} \quad \text{and} \quad p^\ell \leq r < p^{\ell+1}, \quad j \geq 0 \quad \text{and} \quad \ell \geq 1.$$

We have then:

$$\beta_p(t) + \lambda_p(r) = \lambda_p\left(\frac{p}{p-1}t\right) + \lambda_p(r) \leq \lambda_p\left(\frac{p}{p-1}t + r\right) \leq \lambda_p(tr)$$

since

$$tr - \frac{p}{p-1}t - r = (t-1)\left(r - \frac{p}{p-1}\right) - \frac{p}{p-1} \geq$$

$$\geq (p-2)\left(p - \frac{p}{p-1}\right) - \frac{p}{p-1} == ((p-2)^2 - 1)\frac{p}{p-1} \geq 0.$$

2.3. *Let n be a positive integer. Then $\lambda_3(n) \leq \alpha_3(n)$, where:*

$$\alpha_3(n) = [n + 1/4] + \beta_3(n/2).$$

PROOF. We argue by induction on n. The statement is obvious if $n \leq 6$. Let then be $n \geq 7$. We have, by inductive hypothesis, $\lambda_3(n/3) \leq \alpha_3(n/3)$ and so we get:

$$\begin{aligned}\lambda_3(n) &= [n/3] + \lambda_3(n/3) \leq [n/3] + \alpha_3(n/3) = \\ &= [n/3] + [(n+3)/12] + \beta_3(n/2) - [n/4] \leq \\ &\leq \beta_3(n/2) + [n/3] + [n/12] + 1 - [n/4] \leq \beta_3(n/2) + [n/6] + 1 \leq \\ &\leq \beta_3(n/2) + [(n+1)/4] = \alpha_3(n)\end{aligned}$$

2.4. *Let t and m be positive integers. Then:*

$$\beta_p(2m) + \beta_p(t) \leq \lambda_p(p^m t), \quad \textit{for each odd prime } p.$$

PROOF. By **2.2**, it is enough to prove that $\beta_p(2m) \leq \lambda_p(p^m)$. For, since $2m/p - 1 \leq p^{m-1}$, we get:

$$\beta_p(2m) = [2m/p - 1] + \lambda_p(2m/p - 1) \leq p^{m-1} + \lambda_p(p^{m-1}) = \lambda_p(p^m)$$

3. The best estimate for $\alpha_p(n)$

Theorem 3.1. *Let F be a field without primitive cubic roots of unity and let n be an integer ≥ 1. If G is a finite soluble completely reducible subgroup of $GL(n, F)$, then $\alpha_2(G) \leq \lambda_2(n)$.*

PROOF. We argue by induction on n. If G is reducible or irreducible and imprimitive, we argue as in [2], 3. Let then G be irreducible and

primitive. Let A be a maximal normal abelian subgroup of G and let r be the degree of an irreducible A-submodule of $V = V(n, F)$. Let $n = tr$, we separate the cases $r > 1$ and $r = 1$.

1st case: $r > 1$

In this case there exists an extension of F on which $C_G(A)$ possesses a faithful completely reducible representation of degree t and $G/C_G(A)$ is isomorphic to a subgroup of S_r (see [3], p.504). We have $\alpha_2(C_G(A)) \leq \beta_2(2t/3)$ hence $\alpha_2(G) \leq \beta_2(2t/3) + \lambda_2(r)$, from which, by inequality **2.1**, we obtain $\alpha_2(G) \leq \lambda_2(n)$.

2nd case: $r = 1$

In this case $Z(G)$ is the greatest normal abelian subgroup of G and is generated by a scalar transformation. Then $O_3(G) = 1$, since in F there are no primitive cubic roots of unity. It follows that, if N is a minimal normal non-abelian subgroup of G, $N \leq O_q(G)$, with q prime$\neq 3$, and $\exp N = q$, if $q \neq 2$, $\exp N = 4$, if $q = 2$ (see [3], Hilfssatz I). Moreover, $N/Z(N)$ is minimal normal in $G/Z(N)$ and $|N/Z(N)| = q^{2m}$ $(m \geq 1)$; G/C is isomorphic to an irreducible subgroup of the symplectic group $Sp(2m, q)$; $C/C_G(N)$ is isomorphic to a subgroup of $Z(N)^{2m}$, where $C = C_G(N/Z(N))$ (see [3], Hilfssatz II). That being stated, since $q \neq 3$, if 2^ν is the highest power of 2 dividing $|G/C|$, we have $\nu \leq \lambda_2(q^m)$, if $q \neq 2$, $\nu \leq \lambda_2(2m)$, if $q = 2$ (see [7], 1.3 and 1.8 (iv)). We examine separetely the cases $|Z(N)| = q \neq 2$ or $|Z(N)| = 4$ and $|Z(N)| = 2$.

(I) $$|Z(N)| = q \ (q \neq 2) \quad \text{or} \quad |Z(N)| = 4$$

In this case F contains the primitive qth roots or 4th roots of unity; it follows that an irreducible N-submodule of V is absolutely irreducible (and faithful) and has degree q^m (see [3], Hilfssatz III). We have then that $C_G(N)$ possesses a faithful completely reducible representation of degree $t = n/q^m$ over F, hence, by inductive hypothesis, we get $\alpha_2(C_G(N)) \leq \lambda_2(t)$; it follows that, if $q \neq 2$, $\alpha_2(G) \leq \nu + \alpha_2(C) \leq \nu + \alpha_2(C_G(N)) \leq \lambda_2(q^m) + \lambda_2(t) \leq \lambda_2(n)$. If $q = 2$, since $N \leq O_2(C)$, we have $\alpha_2(C) = \alpha_2(C_G(N))$, hence $\alpha_2(G) \leq \nu + \alpha_2(C_G(N)) \leq \lambda_2(2m) + \lambda_2(t) \leq \lambda_2(n)$.

(II) $$|Z(N)| = 2$$

We can assume that G is absolutely irreducible. Otherwise we have $\alpha_2(G) \leq \beta_2(n/3) \leq \lambda_2(n)$. By arguing as in the previous case, we have that $C_G(N)$ possesses a faithful completely reducible representation of degree $t = n/2^m$ over the algebraic closure of F and so $\alpha_2(C_G(N)) \leq \beta_2(2t/3)$; it follows, as in the previous case, $\alpha_2(G) \leq \nu + \alpha_2(C_G(N)) \leq \lambda_2(2m) + \beta_2(2t/3)$, from which, by inequality 2.1, we obtain $\alpha_2(G) \leq \lambda_2(2mt) \leq \lambda_2(n)$.

Theorem 3.2. *Let F be a field of characteristic 2 and let p be an odd prime and n an integer ≥ 1. If G is a finite soluble completely reducible subgroup of $GL(n, F)$, then $\alpha_p(G) \leq \lambda_p(n)$.*

PROOF. We argue by induction on n. Without loss of generality we can assume that F is algebraically closed (see [1], 70.15). If G is reducible or irreducible and imprimitive, we argue as in [2], 3. Let then G be irreducible and primitive. Since F is algebraically closed, the center of G is the greatest normal abelian subgroup of G and is generated by a scalar transformation. Therefore, by arguing as in the proof of the 2nd case of Theorem 3.1 and using the same notation, we have $\alpha_p(G) \leq \nu + \alpha_p(C)$, where p^ν is the highest power of p dividing $|G/C|$. Moreover, since $\mathrm{char}F = 2$, we have $O_2(G) = 1$ and so $q \neq 2$; it follows, as $p \neq 2$, that $\nu \leq \lambda_p(q^m)$ (see [7], 1.3). On the other hand, since F is algebraically closed, there exists a completely reducible $FC_G(N)$-module of degree $t = n/q^m$ on which $C_G(N)$ acts faithfully; we have then, by inductive hypothesis, $\alpha_p(C_G(N)) \leq \lambda_p(t)$. It follows, if $q \neq p$, $\alpha_p(G) \leq \lambda_p(q^m) + \lambda_p(t) \leq \lambda_p(q^m t)$; if $q = p$, we have $N \leq O_p(C)$, hence $\alpha_p(C) = \alpha_p(C_G(N))$; on the other hand, as $O_p(G/C) = 1$, we have $\nu \leq \beta_p(2m)$, hence $\alpha_p(G) \leq \beta_p(2m) + \lambda_p(t)$ and so, by inequality 2.2, we obtain $\alpha_p(G) \leq \lambda_p(2mt) \leq \lambda_p(p^m t) = \lambda_p(n)$.

Theorem 3.3. *Let F be a field without solutions of the equation $x^2 + y^2 = -1$ (and so $\mathrm{char}F = 0$) and let n be an integer≥ 1. If G is a finite soluble subgroup of $GL(n, F)$, then $\alpha_3(G) \leq \alpha_3(n)$, where:*

$$\alpha_3(n) = [n + 1/4] + \beta_3(n/2).$$

PROOF. We argue by induction on n.

Step 1. Suppose that G is reducible. Then G is isomorphic to a subgroup of a direct product $G_1 \times G_2$ where G_i is a finite soluble subgroup of $GL(n_i, F)$ $(i = 1, 2)$ and $n_1 + n_2 = n$. Hence, by inductive hypothesis, we get:

$$\begin{aligned}\alpha_3(G) &\leq \alpha_3(G_1) + \alpha_3(G_2) \leq \\ &\leq [(n_1 + 1)/4] + \beta_3(n_1/2) + [(n_2 + 1)/4] + \beta_3(n_2/2) \leq \\ &\leq [(n + 1)/4] + [n/4] + \sum_{i=1}^{\infty} [n/4\ 3^i] = \alpha_3(n).\end{aligned}$$

Step 2. Suppose that G is irreducible and imprimitive.Then there is a

divisor $t > 1$ of n such that G has a normal subgroup D with the following properties: G/D is isomorphic to a subgroup of S_t; D is isomorphic to a subgroup of a direct product $D_1 \times \cdots \times D_t$ where each D_i is isomorphic to a finite soluble subgroup of $GL(m, F)$ $(m = n/t)$. Hence, by inductive hypothesis, we get:

$$\alpha_3(G) \leq \lambda_3(t) + t[(m+1)/4] + \beta_3(m/2)$$

If $m \leq 3$, the inequality $\alpha_3(G) \leq \alpha_3(n)$ immediately follows. If $m \geq 4$ then $4\ 3^j \leq m < 4\ 3^{j+1}$ $(j \geq 0)$ and

$$\begin{aligned} \alpha_3(G) &\leq \sum_{i=1}^{\infty} [n/m\ 3^i] + t \sum_{i=0}^{j} [m/4\ 3^i] + t[(m+1)/4] \leq \\ &\leq t\big([(m+1)/4] + [m/4]\big) + \sum_{i=1}^{\infty} [n/4\ 3^i] \end{aligned} \tag{3.3.1}$$

The statement obviously follows if $\mathrm{rem}(m, 4) \neq 3$. Otherwise

$$\begin{aligned} &t\big([(m+1)/4] + [m/4]\big) = t(m-1)/2 \leq \\ &\leq [(tm+1)/4] + [tm/4] = [(n+1)/4] + [n/4] \end{aligned}$$

and the statement follows obviously from (3.3.1)

Step 3. Suppose that G is irreducible and primitive. Let A be a maximal normal abelian subgroup of G. If the irreducible A-submodules of $V = V(n, F)$ have degree $r > 1$, there exists an extension of F on which $C_G(A)$ possesses a faithful representation of degree $t = n/r$ and $G/C_G(A)$ is isomorphic to a subgroup of S_r. Hence we have

$$\alpha_3(G) \leq \lambda_3(r) + \beta_3(t)$$

from which, by inequalities 2.2 and 2.3, it follows $\alpha_3(G) \leq \alpha_3(n)$. Finally, we have to consider the case in which the irreducible A-submodules of V have degree 1 and so $A = Z(G)$ is the greatest normal abelian subgroup of G and is generated by a scalar transformation. We can assume that G is absolutely irreducible. Otherwise we have $\alpha_3(G) \leq \beta_3(n/2)$ and therefore $\alpha_3(G) \leq \alpha_3(n)$. Let N be a minimal normal non-abelian subgroup of G and use the same notation as in the proof of the 2nd case of Theorem 3.1. By arguing as in Theorem 3.1, we have that $C_G(N)$ possesses a faithful (completely reducible) representation of degree $t = n/q^m$ over the algebraic closure of F; it follows that $\alpha_3(C_G(N)) \leq \beta_3(t)$ and therefore we get $\alpha_3(G) \leq \nu + \alpha_3(C_G(N)) \leq \nu + \beta_3(t)$, where 3^ν is the highest power of 3 dividing $|G/C|$. If $q \neq 2$ or $q = 2$ and $m > 3$, we have $\nu \leq \lambda_3(q^m)$ (see

[7], 1.3); it follows, by inequalities 2.2 and 2.3, $\alpha_3(G) \leq \lambda_3(q^m) + \beta_3(t) \leq \lambda_3(n) \leq \alpha_3(n)$. Let then be $q = 2$ and $m \leq 3$. Since in F there are no solutions of $x^2 + y^2 = -1$ and therefore there are no primitive 4th roots of unity, we have $|Z(N)| = 2$ and so N is extraspecial; it follows, $N/Z(N)$ being minimal normal in $G/Z(N)$, that, if $m = 1$, N is isomprphic to Q_8 and so, because $x^2 + y^2 = -1$ has no solutions in F, $n > 2$. That being stated, we examine separately the there cases $m = 1, 2, 3$.

(I) $m = 1$. In this case it is $\nu \leq 1$ an so $\nu \leq [n+1/4]$, because $n > 2$. If follows obviously

$$\alpha_3(G) \leq \alpha_3(n).$$

(II) $m = 2$. We have $\nu \leq 2$ and so, since 4 divides n, we get

$$\begin{aligned}\alpha_3(G) &\leq 2 + \beta_3(n/4) \leq 2 + \beta_3(n/2) - 1 = 1 + \beta_3(n/2) \leq \\ &\leq [(n+1)/4] + \beta_3(n/2) = \alpha_3(n).\end{aligned}$$

(III) $m = 3$. We have $\nu \leq 4$ and 8 divides n; it follows

$$\begin{aligned}\alpha_3(G) &\leq 4 + \beta_3(n/8) \leq 4 + \beta_3(n/2) - 2 = 2 + \beta_3(n/2) \leq \\ &\leq [(n+1)/4] + \beta_3(n/2) = \alpha_3(n).\end{aligned}$$

Theorem 3.4. *Let F be a field without solutions of the equation $x^2 + y^2 = -1$ and let n be an integer ≥ 1 and p a Fermat prime $\neq 3$. If G is a finite soluble subgroup of $GL(n, F)$, then $\alpha_p(G) \leq \lambda_p(n)$.*

Proof. We argue by induction on n. If G is reducible or irreducible and imprimitive, we argue as in [2], 3. Let the G be irreducible and primitive. Let A be a maximal normal abelian subgroup of G and let r be the degree of an irreducible A-submodule of $V = V(n, F)$. Let $n = tr$, we separate the cases $r > 1$ and $r = 1$.

1st case: $r > 1$

In this case, by arguing as in the 1st case of Theorem 3.1, we obtain $\alpha_p(G) \leq \lambda_p(r) + \beta_p(t)$, from which, by inequality 2.2, we get $\alpha_p(G) \leq \lambda_p(n)$.

2nd case: $r = 1$

We can assume that G is absolutely irreducible. Otherwise we have $\alpha_p(G) \leq \beta_p(n/2) \leq \lambda_p(n)$. Let N be a minimal normal non-abelian subgroup of G and use the same notation as in the proof of the 2nd case of Theorem 3.1. By arguing as in Theorem 3.1, we have that $C_G(N)$ possesses a faithful (completely reducible) representation of degree $t = n/q^m$ on the algebraic closure of F; it follows that $\alpha_p(C_G(N)) \leq \beta_p(t)$ and therefore $\alpha_p(G) \leq \nu + \beta_p(t)$, where p^ν is the highest power of p dividing

$|G/C|$. If $q = p$, we have $O_p(G/C) = 1$, hence $\nu \leq \beta_p(2m)$; it follows $\alpha_p(G) \leq \beta_p(2m) + \beta_p(t)$ and the statement $\alpha_p(G) \leq \lambda_p(n)$ follows from inequality 2.4. Let then $q \neq p$. If $q \neq 2$ or $q = 2$ and $p \neq 2^m + 1$, we have, as $p \neq 3$, $\nu \leq \lambda_p(q^m)$ (see [7], 1.3); it follows $\alpha_p(G) \leq \lambda_p(q^m) + \beta_p(t)$, from which, by inequality 2.2, we get $\alpha_p(G) \leq \lambda_p(n)$. If $q = 2$ and $p = 2^m + 1$, we have $\nu \leq 1$, hence $\alpha_p(G) \leq 1 + \beta_p(t)$, and so, as $p \neq 3$, we get $\alpha_p(G) \leq \lambda_p(n)$, if $t \geq 2$. Let finally be $q = 2$, $p = 2^m + 1$, $t = 1$ and $\alpha_p(G) = 1$. Since in F there are no primitive 4th roots of unity, N is extraspecial. On the other hand, as p divides $|G/C_G(N)|$, N possesses an automorphism σ of order p; it follows that σ acts irreducibly on $N/Z(N)$ and so tha Witt index of the quadratic from of N is not maximal (see, for instance, [4], 13.9); but such a group N has no faithful representation of degree $p-1$ on a field without solutions of the equation $x^2 + y^2 = -1$ (see, for instance, [3], Hilfssatz III).

Remark. Proposition 1.4 assures that the estimate provided for $\alpha_p(n)$ in Theorems 3.1, 3.2, and 3.4 are the best possible. Also, the estimate provided for $\alpha_3(n)$ in Theorem 3.3 is the best possible. To prove that, it is enough (Lemma 1.1) to find a finite soluble subgroup G_0 of $GL(4, Q)$ with $\alpha_3(G_0) = 2$ (Q the rational field). Let G_0 be the group of left and right multiplications with unit quaternions as linear transformations of the 4-space of quaternions. G_0 is the holomorph of $Q_8 \dot{\times} Q_8$ by an elementary abelian group of order 9 of its authomorphisms; thus $\alpha_3(G_0) = 2$.

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