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# Degree of approximation to functions in a normed space

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**1.** Let $\{s_n\}$ be the sequence of partial sums of $\sum a_n$. Let $\Lambda = (\lambda_{nk})$ be a lower triangular infinite matrix, i.e. $\lambda_{nk} = 0$ for $k > n$ and let $\Lambda$–transform of the sequence $\{s_n\}$ be given by

$$T_n = \sum_{k=0}^{n} \lambda_{nk} s_k, \quad n = 0, 1, 2, \dots . \tag{1.1}$$

Let $C_{2\pi}$ be the class of all $2\pi$–periodic continuous functions $f$ on $[0, 2\pi]$ having Fourier series

$$f \sim a_0/2 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx). \tag{1.2}$$

We define the space $H_\omega$ by

$$H_\omega = \{f \in C_{2\pi} : |f(x) - f(y)| \le K\omega(|x-y|)\} \tag{1.3}$$

and the norm $\|\,.\,\|_{\omega^*}$ by

$$\|f\|_{\omega^*} = \|f\|_C + \sup_{x,y}\{\Delta^{\omega^*} f(x,y)\}, \tag{1.4}$$

where

$$\|f\|_C = \sup_{0 \le x \le 2\pi} |f(x)|, \tag{1.5}$$

and

$$\Delta^{\omega^*} f(x,y) = \frac{|f(x) - f(y)|}{\omega^*(|x-y|)}, \quad x \ne y, \tag{1.6}$$

and $\Delta^0 f(x,y) = 0$, $\omega(t)$ and $\omega^*(t)$ being increasing functions of $t$. If $\omega(|x-y|) \le A|x-y|^\alpha$ and $\omega^*(|x-y|) \le K|x-y|^\beta$, $0 \le \beta < \alpha \le 1$, $A$

and $K$ being positive constants, then the space

$$H_\alpha = \{f \in C_{2\pi} : |f(x) - f(y)| \leq K|x-y|^\alpha,\ 0 < \alpha \leq 1\} \tag{1.7}$$

is a Banach space ([4]) and the metric induced by the norm $\|\cdot\|_\alpha$ on $H_\alpha$ is said to be a Hölder metric. We write

$$\emptyset_x(t) = f(x+t) + f(x-t) - 2f(x). \tag{1.8}$$

Throughout the paper $f$ will be taken to be periodic and $\omega(t)$ the modulus of continuity of $f \in C_{2\pi}$. $\{P_n\}\updownarrow$ and $\{p_n\}\updownarrow$ stands for non-decreasing and non-increasing sequences, respectively.

The following theorem is proved by CHANDRA ([2]) taking sup norm $\|\cdot\|$ on $0 \leq x \leq 2\pi$.

**Theorem.** *Let $\Lambda = (\lambda_{nk})$ satisfy the following conditions.*

$$\lambda_{nk} \geq 0 \qquad (n,k = 0,1,2,\dots),\ \sum_{k=0}^{n} \lambda_{nk} = 1, \tag{1.9}$$

*and*

$$\lambda_{nk} \leq \lambda_{n,k+1} \qquad (k = 0,1,2,\dots,n-1;\ n = 0,1,2,\dots). \tag{1.10}$$

*Let* $\omega(t) \mid \int^{\pi} u^{-2}\omega(u)\,du = 0(H(t)),$

$$\int_t u^{-2}\omega(u)\,du = 0(H(t)), \quad H(t) \geq 0, \tag{1.11}$$

*and*

$$\int_0^t H(u)\,du = 0(tH(t)),\ \text{as } t \to 0+. \tag{1.12}$$

*Then*

$$\|T_n(f,x) - f\| = 0(\omega(\pi/n)) + 0(\lambda_{nn}H(\pi/n)). \tag{1.13}$$

*If, in addition to* (1.11), (1.12) *be satisfied, then*

$$\|T_n(f,x) - f\| = 0(\lambda_{nn}(H(\pi/n)). \tag{1.14}$$

**2.** The object of this paper is to widen the scope of the above theorem of CHANDRA under more general assumptions and to include a number of interesting results. For analogous conditions on the function, one may refer to MOHAPATRA and CHANDRA ([3]) and SINGH ([6]). Precisely, we prove

**Theorem I.** *Let $\Lambda = (\lambda_{nk})$ satisfy the conditions* (1.9) *and* (1.10). *Then for $f \in H_\omega$, $0 \le \beta < \eta \le 1$*

$$\begin{aligned}\|T_n(f,x) - f\|_{\omega^*} = 0[\{\omega(|x-y|)\}^{\beta/\eta}\{\omega^*(|x-y|)\}^{-1}\cdot \\ \cdot (H(\pi/n))^{1-\beta/\eta}\lambda_{nn}\{n^{\beta/\eta} + \lambda_{nn}^{-\beta/\eta}\}] + 0(\lambda_{nn}H(\pi/n)),\end{aligned} \tag{2.1}$$

*if $\omega(t)$ satisfies* (1.11) *and* (1.12), *and*

$$\begin{aligned}\|T_n(f,x) - f\|_{\omega^*} = 0[\{\omega(|x-y|)\}^{\beta/\eta}\{\omega^*(|x-y|)\}^{-1}\cdot \\ \cdot \{(\omega(\pi/n))^{1-\beta/\eta} + \lambda_{nn}n^{\beta/\eta}(H(\pi/n))^{1-\beta/\eta}\}] + \\ + 0(\omega(\pi/n)) + 0(\lambda_{nn}H(\pi/n)),\end{aligned} \tag{2.2}$$

*if $\omega(t)$ satisfies* (1.11).

**Theorem II.** *Let $\Lambda = (\lambda_{nk})$ satisfy the conditions* (1.9) *and*

$$\lambda_{nk} \ge \lambda_{n,k+1}\ (k = 0,1,2,\dots,n-1;\ n = 0,1,2,\dots). \tag{2.3}$$

*Let $\omega(t)$ be such that the conditions* (1.11) *and* (1.12) *be satisfied, then for $f \in H_\omega$, $0 \le \beta < \eta \le 1$*

$$\begin{aligned}\|T_n(f,x) - f\|_{\omega^*} = 0[\{\omega(|x-y|)\}^{\beta/\eta}\{\omega^*(|x-y|)\}^{-1}\cdot \\ \cdot \lambda_{n0}(H(\lambda_{n0}))^{1-\beta/\eta}\{n^{\beta/\eta} + \lambda_{n0}^{-\beta/\eta}\}] + 0(\lambda_{n0}H(\lambda_{n0})).\end{aligned} \tag{2.4}$$

**3.** We shall require the following lemma in the proof of the theorems.

**Lemma.** *Let $\omega(t)$ satisfy* (1.11) *and* (1.12), *then*

$$\int_0^u t^{-1}\omega(t)\,dt = 0(uH(u)), \quad u \to 0+. \tag{3.1}$$

For the proof of the lemma see CHANDRA ([2]).

**4. Proof of Theorem I.** It is to be noted that

$$|\emptyset_x(t) - \emptyset_y(t)| \le 4K\omega(|t|), \tag{4.1}$$

and also

$$|\emptyset_x(t) - \emptyset_y(t)| \le 4A\omega(|x-y|). \tag{4.2}$$

We have

$$T_n(f,x) = \sum_{k=0}^{n} \lambda_{n,k}\ S_k(x).$$

Setting

$$E_n(x) = T_n(f,x) - f(x) = \frac{1}{2\pi}\int_0^{\pi} \frac{\emptyset_x(t)}{\sin(t/2)} \sum_{k=0}^{n} \lambda_{nk} \sin(k+1/2)t\ dt$$

and

$$\begin{aligned} E_n(x,y) &= E_n(x) - E_n(y) = \\ &= \frac{1}{2\pi}\int_0^{\pi} \frac{\emptyset_x(t) - \emptyset_y(t)}{\sin(t/2)} \sum_{k=0}^{n} \lambda_{nk} \sin(k+1/2)t\ dt, \end{aligned}$$

we get

$$|E_n(x,y)| \leq \int_0^{\pi/n} + \int_{\pi/n}^{\pi} = I_1 + I_2, \tag{4.3}$$

say, where

$$I_1 = \frac{1}{2\pi}\int_0^{\pi/n} \frac{|\emptyset_x(t) - \emptyset_y(t)|}{\sin(t/2)} \Big| \sum_{k=0}^{n} \lambda_{nk} \sin(k+1/2)t \Big|\, dt$$

and

$$I_2 = \frac{1}{2\pi}\int_{\pi/n}^{\pi} \frac{|\emptyset_x(t) - \emptyset_y(t)|}{\sin(t/2)} \Big| \sum_{k=0}^{n} \lambda_{nk} \sin(k+1/2)t \Big|\, dt.$$

Now using (1.9), (4.1) and the lemma and taking $\sin(k+1/2)t = 0(1)$, we get

$$I_1 = 0(1)\int_0^{\pi/n} t^{-1}\omega(t)\ dt = 0\left(\frac{\pi}{n} H(\pi/n)\right) = 0(\lambda_{nn} H(\pi/n)), \tag{4.4}$$

since by (1.10), $\lambda_{nk} \leq \lambda_{n,k+1}$ and, therefore, $(n+1)\lambda_{nn} \geq \sum\limits_{k=0}^{n} \lambda_{nk} = 1$ gives $n^{-1} = 0(\lambda_{nn})$. Further, by (1.11) and (1.12) and integration by parts

$$I_1 = 0(1)\int_0^{\pi/n} t^{-1}\omega(t)\sum_{k=0}^{n}\lambda_{nk}(k+1/2)t dt = 0(n)\int_0^{\pi/n}\omega(t)dt \tag{4.5}$$

$$= 0(n)\left[-t^2\int_t^{\pi} u^{-2}\omega(u)du\Big|_0^{\pi/n} + \int_0^{\pi/n} 2t dt\int_t^{\pi} u^{-2}\omega(u)du\right]$$

$$= 0\left[n^{-1}H(\pi/n)\right] = 0[\lambda_{nn}H(\pi/n)]. \tag{4.6}$$

But from (4.5) and (1.12)

$$I_1 = 0(n)\,\omega(\pi/n)\cdot\frac{\pi}{n} = 0(\omega(\pi/n)). \tag{4.7}$$

By (1.10) and Abel's lemma, we see that

$$\sum_{k=0}^{n}\lambda_{nk}\sin(k+1/2)t = 0(\lambda_{nn}t^{-1}) \tag{4.8}$$

and, therefore, by (1.11)

$$I_2 = 0(\lambda_{nn})\int_{\pi/n}^{\pi} t^{-2}\omega(t)\,dt = 0(\lambda_{nn}H(\pi/n)). \tag{4.9}$$

Again by (1.9) and (4.2)

$$\begin{aligned} I_1 &= 0[\omega(|x-y|)]\int_0^{\pi/n} t^{-1}\sum_{k=0}^{n}\lambda_{nk}|\sin(k+1/2)t|\;dt \\ &= 0[\omega(|x-y|)]\int_0^{\pi/n}\sum_{k=0}^{n}\lambda_{nk}(k+1/2)\,dt \\ &= 0[\omega(|x-y|)], \end{aligned} \tag{4.10}$$

and by (4.2) and (4.8)

$$I_2 = 0[\omega(|x-y|)]\int_{\pi/n}^{\pi} t^{-2}\lambda_{nn}\,dt = 0[\omega(|x-y|)\,n\,\lambda_{nn}]. \tag{4.11}$$

Now noting that

$$I_r = I_r^{1-\beta/\eta}\,I_r^{\beta/\eta},\quad r = 1,2\;, \tag{4.12}$$

we have, from (4.4) or (4.6) and (4.10)

$$
\begin{aligned}
I_1 &= [0\{\lambda_{nn} H(\pi/n)\}]^{1-\beta/\eta}\ [0\{\omega(|x-y|)\}^{\beta/\eta}] \\
&= 0[\{\omega(|x-y|)\}^{\beta/\eta}\ \{\lambda_{nn} H(\pi/n)\}^{1-\beta/\eta}],
\end{aligned}
\tag{4.13}
$$

and from (4.9) and (4.11)

$$
I_2 = 0[\{\omega(|x-y|)\}^{\beta/\eta}\ \lambda_{nn}\ n^{\beta/\eta}\{H(\pi/n)\}^{1-\beta/\eta}].
\tag{4.14}
$$

On the other hand, from (4.7) and (4.10)

$$
I_1 = 0[\{\omega(|x-y|)\}^{\beta/\eta}\ \{\omega(\pi/n)\}^{1-\beta/\eta}],
\tag{4.15}
$$

Thus from (4.13) and (4.14)

$$
\begin{aligned}
\sup_{x,y} |\Delta^{\omega^*} E_n(x,y)| &= \sup_{x,y} \frac{|E_n(x) - E_n(y)|}{\omega^*(|x-y|)} \\
&= 0[\{\omega^*(|x-y|)\}^{-1}\{\omega(|x-y|)\}^{\beta/\eta} \cdot \\
&\quad \cdot \left\{H\left(\frac{\pi}{n}\right)\right\}^{1-\beta/\eta} \lambda_{nn}\{\lambda_{nn}^{-\beta/\eta} + n^{\beta/\eta}\}],
\end{aligned}
\tag{4.16}
$$

and from (4.14) and (4.15)

$$
\begin{aligned}
&= 0[\{\omega^*(|x-y|)\}^{-1}\{\omega(|x-y|)\}^{\beta/\eta} \cdot \\
&\quad \{(\omega(\pi/n))^{1-\beta/\eta} + \lambda_{nn} n^{\beta/\eta} \cdot \\
&\quad \cdot (H(\pi/n))^{1-\beta/\eta}\}].
\end{aligned}
\tag{4.17}
$$

It is to be noted that

$$
\begin{aligned}
\|E_n(x)\|_C &= \max_{0 \le x \le 2\pi} |T_n(f,x) - f| \\
&= \begin{cases} 0(\lambda_{nn} H(\pi/n)), & \text{from (4.6) and (4.9)} \\ 0(\omega(\pi/n)) + 0(\lambda_{nn} H(\pi/n)), & \text{from (4.7) and (4.9).} \end{cases}
\end{aligned}
\tag{4.18}
$$

Combining (4.16) with the first part of (4.18) and (4.17) with the second part of (4.18), the required results are established.

**5. Proof of Theorem II.** From the proof of Theorem I, we have

$$
|E_n(x,y)| \le \int_0^{\lambda_{n0}} + \int_{\lambda_{n0}}^{\pi} = J_1 + J_2.
\tag{5.1}
$$

By the lemma

$$J_1 \le \int_0^{\lambda_{n0}} t^{-1}\omega(t) \sum_{k=0}^{n} \lambda_{nk} \, |\sin(k+1/2)t| \, dt = 0(\lambda_{n0} H(\lambda_{n0})), \tag{5.2}$$

and by (1.9), Abel's lemma and (2.3), we have

$$J_2 = 0(\lambda_{n0}) \int_{\lambda_{n0}}^{\pi} t^{-2}\omega(t)\, dt = 0(\lambda_{n0} H(\lambda_{n0})). \tag{5.3}$$

Again, using (4.2) and (1.9), we get

$$J_1 = 0(n\ \omega(|x-y|)) \sum_{k=0}^{n} \lambda_{nk} \int_0^{\lambda_{n0}} dt = 0(n\ \omega(|x-y|)\lambda_{n0}), \tag{5.4}$$

and

$$J_2 = 0(\omega(|x-y|) \int_{\lambda_{n0}}^{\pi} \lambda_{n0}\ t^{-2} dt = 0(\omega(|x-y|). \tag{5.5}$$

Proceeding as in the proof of theorem I, we have

$$J_1 = 0\left[\lambda_{n0}\{n\omega(|x-y|\}^{\beta/\eta}\{H(\lambda_{n0})\}^{1-\beta/\eta}\right], \tag{5.6a}$$

and

$$J_2 = 0\left[\{(\omega(|x-y|\}^{\beta/\eta}\{H(\lambda_{n0})\lambda_{n0}\}^{1-\beta/\eta}\right]. \tag{5.6b}$$

Thus

$$\begin{aligned} \operatorname*{Sup}_{x,y} |\Delta^{\omega^*} E_n(x,y)| = 0\Big[&\{\omega(|x-y|)\}^{\beta/\eta}\{\omega^*(|x-y|)\}^{-1} \cdot \\ &\cdot \lambda_{n0}\{H(\lambda_{n0})\}^{1-\beta/\eta}\{n^{\beta/\eta} + \lambda_{n0}^{-\beta/\eta}\}\Big]. \end{aligned} \tag{5.7}$$

It is obvious from (5.2) and (5.3) that

$$\|E_n(x,y)\|_C = 0(\lambda_{n0} H(\lambda_{n0})). \tag{5.8}$$

Combine (5.7) and (5.8) for the proof of the Theorem II.

**6. Corollaries.** The following are the corollaries based on Theorem I.

Put $\omega^*(|x-y|) \le K|x-y|^\beta,\ \omega(|x-y|) \le A|x-y|^\alpha,\ H(t) = \log(\pi/t)$ for $\alpha = 1$ and $H(t) = t^{\alpha-1}$ for $0 < \alpha < 1$ and replace $\eta$ by $\alpha$ to get

**Corollary 1.** *Let* $\Lambda = (\lambda_{nk})$ *satisfy* (1.9) *and* (1.10) *and let* $f \in H_\alpha,\ 0 \le \beta < \alpha \le 1$, *then*

$$\|T_n(f,x) - f\|_\beta = \begin{cases} 0(\lambda_{nn}\log n) + 0\left[n^\beta\lambda_{nn}(\log n)^{1-\beta} + (\lambda_{nn}\log n)^{1-\beta}\right], \\ \qquad\qquad \alpha = 1 \\ 0\left[n^{1-\alpha+\beta}\lambda_{nn}\right] + 0\left[(\lambda_{nn}n^{1-\alpha})^{1-\beta/\alpha}\right],\ 0 < \alpha < 1, \end{cases}$$

and

$$\|T_n(f,x) - f\|_\beta = \begin{cases} 0(n^{\beta-1}) + 0(\lambda_{nn}\ n^\beta(\log n)^{1-\beta}), & \alpha = 1, \\ 0(n^{\beta-\alpha}) + 0(\lambda_{nn}\ n^{1-\alpha+\beta}), & 0 < \alpha < 1. \end{cases}$$

The later result is the theorem 2 due to MOHAPATRA and CHANDRA ([3]).

If we put $\beta = 0$, so that $\omega^*(|x-y|) \le K$ and the norm $\|\,.\,\|$ stands for the sup norm on $C_{2\pi}$ in Theorem I, then it reduces to the CHANDRA's Theorem (loc. cit.).

If $\beta = 0$ and $f \in \operatorname{Lip}\alpha,\ 0 < \alpha \le 1$, so that $\omega(t) = 0(t^\alpha)$ and

$$H(t) = \begin{cases} \log(\pi/t), & \alpha = 1, \\ t^{\alpha-1}, & 0 < \alpha < 1, \end{cases} \tag{6.1}$$

we have from Corollary 1

**Corollary 2.** *Let* $\Lambda = (\lambda_{nk})$ *satisfy* (1.9), (1.10) *and* $f \in \operatorname{Lip}\alpha,\ 0 < \alpha \le 1$, *then*

$$\|T_n(f,x) - f\| = \begin{cases} 0(\lambda_{nn}\ \log n), & \alpha = 1, \\ 0(n^{1-\alpha}\lambda_{nn}), & 0 < \alpha < 1, \end{cases}$$

and

$$\|T_n(f,x) - f\| = \begin{cases} 0(n^{-1}) + 0(\lambda_{nn}\ \log n), & \alpha = 1, \\ 0(n^{-\alpha}) + (n^{1-\alpha}\lambda_{nn}), & 0 < \alpha < 1. \end{cases}$$

If we put $\omega^*(|x-y|) \le K|x-y|^\beta,\ \omega(|x-y|) \le A|x-y|^\alpha$ and replace $\eta$ by $\alpha$ in Theorem II, we have by (6.1).

**Corollary 3.** *Let* $\Lambda = (\lambda_{nk})$ *satisfy* (1.9) *and* (2.3) *and let* $f \in H_\alpha$, $0 \le \beta < \alpha \le 1$, *then*

$$\|T_n(f,x) - f\|_\beta = \begin{cases} 0\left[\left(\lambda_{n0}\log\frac{\pi}{\lambda_{n0}}\right)\right] + 0\left[n^\beta\lambda_{n0}\left(\log\frac{\pi}{\lambda_{n0}}\right)^{1-\beta}\right] + \\ \qquad +0\left[\left(\lambda_{n0}\ \log\frac{\pi}{\lambda_{n0}}\right)^{1-\beta}\right], & \alpha = 1, \\ 0(\lambda_{n0}^\alpha) + 0(n^{\beta/\alpha}\lambda_{n0}^{\alpha-\beta+\beta/\alpha}) + 0(\lambda_{n0}^{\alpha-\beta}), & 0 < \alpha < 1. \end{cases}$$

If we put $\beta = 0$ in the above corollary, then

**Corollary 4.** *Let* $\Lambda = (\lambda_{nk})$ *satisfy* (1.9) *and* (2.3) *and* $f \in \operatorname{Lip}\alpha$, $0 < \alpha \le 1$, *then*

$$\|T_n(f,x) - f\| = \begin{cases} 0\left(\lambda_{n0}\log\frac{\pi}{\lambda_{n0}}\right), & \alpha = 1, \\ 0(\lambda_{n0}^\alpha), & 0 < \alpha < 1. \end{cases}$$

Now we specialize the matrix $\Lambda = (\lambda_{nk})$. If we put $\lambda_{nk} = \frac{p_k}{P_n}$ and $\lambda_{nk} = \frac{p_{n-k}}{P_n}$ in the transform $T_n(f,x)$, we have, respectively

$$\overline{N}_n(f,x) = \frac{1}{P_n}\sum_{k=0}^{n} p_k S_k(x),$$

and

$$N_n(f,x) = \frac{1}{P_n}\sum_{k=0}^{n} p_{n-k} S_k(x),$$

transforms, where

$$P_n = \sum_{k=0}^{n} p_k, \quad p_n > 0.$$

If we put $\omega^*(|x-y|) \le K|x-y|^\beta$, $\omega(|x-y|) \le A|x-y|^\alpha$, and replace $\eta$ by $\alpha$ in Theorems I and II, and if we put $\lambda_{nn} = p_n/P_n$, $\{p_n\}\updownarrow$ in Theorem I and $\lambda_{n0} = p_n/P_n$, $\{P_n\}\updownarrow$, in Theorem II, then the following corollaries are obtained, respectively, keeping in view (6.1)

**Corollary 6.** *For* $f \in H_\alpha$, $0 \le \beta < \alpha \le 1$, $\{p_n\}\updownarrow$, *we have*

$$\|\overline{N}_n(f,x) - f\|_\beta = \begin{cases} 0\left(\frac{p_n}{P_n}n^{1-\alpha+\beta}\right) + 0\left(n^{1-\alpha}\frac{p_n}{P_n}\right)^{1-\beta/\alpha}, & 0 < \alpha < 1, \\ 0\left(n^\beta\frac{p_n}{P_n}(\log n)^{1-\beta}\right), & \alpha = 1. \end{cases}$$

*and*

$$\|\overline{N}_n(f,x) - f\|_\beta = \begin{cases} 0(n^{\beta-\alpha}) + 0\left(\frac{p_n}{P_n} n^{1-\alpha+\beta}\right), & 0 < \alpha < 1, \\ 0(n^{\beta-1}) + 0\left(\frac{p_n}{P_n} n^{\beta} \log^{1-\beta} n\right), & \alpha = 1. \end{cases}$$

**Corollary 7.** *For $f \in H_\alpha,\ 0 \le \beta < \alpha \le 1,\ \{p_n\}\updownarrow$, we have*

$$\|N_n(f,x) - f\|_\beta = \begin{cases} 0\left[n^\beta \frac{p_n}{P_n}\left(\log\frac{P_n}{p_n}\right)^{1-\beta}\right] + 0\left[\left(\frac{p_n}{P_n}\log\frac{P_n}{p_n}\right)^{1-\beta}\right], \\ \qquad \alpha = 1 \\ 0\left(\left(\frac{p_n}{P_n}\right)^{\alpha-\beta}\right) + 0\left[n^{\beta/\alpha}\left(\frac{p_n}{P_n}\right)^{\alpha-\beta+\beta/\alpha}\right], \\ \qquad 0 < \alpha < 1. \end{cases}$$

Putting $\beta = 0$ in the Corollary 6, we have special results for $\overline{N}_n(f,x)$ transformations and from Corollary 7, we have the following theorem due to SINGH ([5]).

**Corollary 8.** *If $f \in \mathrm{Lip}\,\alpha,\ 0 < \alpha \le 1$ and $\{p_n\}\updownarrow$, then*

$$\|N_n(f,x) - f\| = \begin{cases} 0\left[\frac{p_n}{P_n}\log\frac{P_n}{p_n}\right), & \alpha = 1, \\ 0\left[\left(\frac{p_n}{P_n}\right)^\alpha\right], & 0 < \alpha < 1. \end{cases}$$

If we put $\omega^*(|x-y|) \le K|x-y|^\beta,\ \omega(|x-y|) \le A|x-y|^\alpha,\ \lambda_{nn} = 1/n$ and replace $\eta$ by $\alpha$ in the second part of Theorem I, we have

**Corollary 9.** *For $f \in H_\alpha,\ 0 \le \beta < \alpha \le 1$*

$$\|\sigma_n(f,x) - f\|_\beta = \begin{cases} 0(n^{\beta-\alpha}), & 0 < \alpha < 1, \\ 0[n^{\beta-1}(1+\log n)^{1-\beta}], & \alpha = 1. \end{cases}$$

This is result of PRÖSSDORF ([4]), where $\sigma_n(f,x)$ is the Fejér operator. Put $\beta = 0$ in the above corollary to get one of the results of ALEXITS ([1]).

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