

Characterization of additive functions with values in a compact Abelian group

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1. Let G be an additively written, metrically compact Abelian topological group, the one-dimensional torus. Let $\mathbf{N}$ denote the set of all positive integers. A function $\phi : \mathbf{N} \rightarrow G$ is called completely additive, if

$$\phi(nm) = \phi(n) + \phi(m)$$

holds for each couple $n, m \in \mathbf{N}$. Let A_G^* be the class of completely additive functions.

Let $A > 0, B$ be fixed integers. We shall say that an infinite sequence $\{x_\nu\}_{\nu=1}^\infty$ in G is of property $D[A, B]$ if for any convergent subsequence $\{x_{\nu_n}\}_{n=1}^\infty$ the sequence $\{x_{A\nu_n+B}\}_{n=1}^\infty$ has a limit, too. We say that it is of property $E[A, B]$ if for any convergent subsequence $\{x_{A\nu_n+B}\}_{n=1}^\infty$ the sequence $\{x_{\nu_n}\}_{n=1}^\infty$ is convergent. We shall say that an infinite sequence $\{x_\nu\}_{\nu=1}^\infty$ in G is of property $\Delta[A, B]$ if $\{x_{A\nu+B} - x_\nu\}_{\nu=1}^\infty$ is convergent.

Let $A_G^*(D[A, B])$, $A_G^*(E[A, B])$ and $A_G^*(\Delta[A, B])$ be the classes of those $\phi \in A_G^*$ for which $\{x_\nu = \phi(\nu)\}_{\nu=1}^\infty$ is of property $D[A, B]$, $E[A, B]$ and $\Delta[A, B]$, respectively.

It is obvious that

$$A_G^*(\Delta[A, B]) \subseteq A_G^*(D[A, B])$$

and

$$A_G^*(\Delta[A, B]) \subseteq A_G^*(E[A, B]).$$

Z. DARÓCZY and I. KÁTAI [1] proved in the case $A = B = 1$ that

$$A_G^*(\Delta[1, 1]) = A_G^*(D[1, 1]).$$

By using an unpublished result due to E. WIRSING [7] which asserts that $\phi \in A_G^*(\Delta[1, 1])$ if and only if

$$\phi(n) \equiv \tau \log n \pmod{1} \quad (\forall n \in \mathbf{N})$$

for some real number τ , Z. DARÓCZY and I. KÁTAI [2] deduced the following assertion: If $\phi \in A_G^*(\Delta[1, 1]) = A_G^*(D[1, 1])$, then there exists a continuous homomorphism $\psi: \mathbf{R}_x \rightarrow G$, where $\mathbf{R}_x$ denotes the multiplicative group of the positive reals, such that ϕ is a restriction of ψ on the set $\mathbf{N}$, i.e.

$$\phi(n) = \psi(n) \quad (\forall n \in \mathbf{N}).$$

For the case $A = 2$ and $B = -1$ the complete characterization of $A_G^*(D[2, -1])$ and $A_G^*(\Delta[2, -1])$ has been given by Z. DARÓCZY and I. KÁTAI [3], [4]. The basic idea of their proof is to reduce the condition $\phi \in A_G^*(D[2, -1])$ to the relation

$$\phi(2n+1) - \phi(2n-1) \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

and apply a modification of Wirsing's theorem.

Our main purpose in this paper is to give a complete determination of $A_G^*(E[A, B])$ and of $A_G^*(\Delta[A, B])$. We shall prove the following

Theorem 1. For any fixed integers $A > 0$ and $B \neq 0$, we have $A_G^*(E[A < B]) = A_G^*(\Delta[A, B])$.

Theorem 2. Let $A > 0$, $B \neq 0$ be fixed integers. If

$$\phi \in A_G^*(E[A, B]) = A_G^*(\Delta[A, B])$$

then there exists a continuous homomorphism $\psi: \mathbf{R}_x \rightarrow G$, where $\mathbf{R}_x$ denotes the multiplicative group of the positive reals, such that ϕ is a restriction of ψ on the set $\mathbf{N}$, i.e.

$$\phi(n) = \psi(n)$$

for all $n \in \mathbf{N}$.

Conversely, let $\psi: \mathbf{R}_x \rightarrow G$ be an arbitrary continuous homomorphism. Then the function

$$\phi(n) := \psi(n) \quad (n = 1, 2, \dots)$$

belongs to $A_G^*(E[A, B]) = A_G^*(\Delta[A, B])$.

2. PROOF OF THEOREM 1.

Assume that $A > 0$ and $B \neq 0$ are integers.

Let $\phi \in A_G^*(E[A, B])$. Let X denote the set of limit points of $\{\phi(n) \mid n \in \mathbf{N}\}$, i.e. $g \in X$ if there exists a sequence

$$n_1 < \dots < n_\nu < \dots \quad (n_\nu \in \mathbf{N}),$$

for which $\phi(n_\nu) \rightarrow g$. Let $X_1 (\subseteq X)$ be the set of limit points of $\{\phi(An+1) \mid n \in \mathbf{N}\}$. Since $\mathbf{N}$ and the natural numbers $m \equiv 1 \pmod{A}$ form semigroups, therefore X and X_1 are semigroups as well. Thus, X and X_1 are closed

semigroups in G , so by a known theorem (see [6], Theorem (9.16)) they are compact groups. Since $0 \in X_1 \subseteq X$, we have

$$(2.1) \quad \phi(n) \in X \text{ and } \phi(An + 1) \in X_1 \text{ for each } n \in \mathbb{N}.$$

Let X_B denote the set of limit points of $\{\phi(An + B) \mid n \in \mathbb{N}\}$. If $g \in X_B$, then there is a sequence $\{n_\nu\}_{\nu=1}^\infty$ such that $\phi(An_\nu + B) \rightarrow g$. Since $\phi \in A_G^*(E[A, B])$, the sequence $\{\phi(n_\nu)\}_{\nu=1}^\infty$ is convergent. Let $\phi(n_\nu) \rightarrow g'$. It is obvious that g' is determined by g , and so the correspondence $F: g \rightarrow g'$ is a function, furthermore $F(X_B) = X$. For the proof of these simple assertions see [1].

Lemma 1. We have

$$F(g) = g - \phi(A)$$

for every $g \in X_B$.

PROOF. Since X_1 is a subgroup in G , we have $0 \in X_1$, and so there exists a sequence

$$N_1 < \dots < N_\nu < \dots \quad (N_\nu \in \mathbb{N})$$

for which $\phi(AN_\nu + 1) \rightarrow 0$. Since G is sequentially compact, therefore $\{\phi(N_\nu)\}_{\nu=1}^\infty$ contains at least one limit point. Let

$$(2.2) \quad \phi(N_{\nu_k}) \rightarrow \tau \quad (\tau \in X).$$

Let $g \in X_B$ be an arbitrary element. By using (2.1) we have $\phi(A) \in X$. Since $g \in X_B \subseteq X$ and X is a group, we have $g - \phi(A) \in X$. Thus, it follows from $F(X_B) = X$ that there exists an element $h \in X_B$, for which

$$(2.3) \quad F(h) = g - \phi(A).$$

From the definition of X_B it is clear that there exists a sequence

$$M_1 < \dots < M_\nu < \dots \quad (M_\nu \in \mathbb{N})$$

for which $\phi(AM_\nu + B) \rightarrow h$.

Let us consider the sequence

$$\{\phi(A^2 M_{\nu_k} N_{\nu_k} + B)\}_{k=1}^\infty,$$

where ν_k is determined in (2.2). Since G is sequentially compact, therefore the above sequence contains at least one limit point. Let

$$(2.4) \quad \phi(A^2 M_{\nu_{k_j}} N_{\nu_{k_j}} + B) \rightarrow h' \quad (h' \in X_B).$$

From the definition of F it follows by (2.2) and (2.3) that

$$(2.5) \quad F(h') = g + \tau.$$

Applying the following relation

$$(A^2mn + B)(An + 1) = An[Am(An + 1) + B] + B$$

with $m = M_{\nu_{k_j}}, n = N_{\nu_{k_j}}$ and using the definition of F and (2.4), we have

$$\begin{aligned}\phi\{AN_{\nu_{k_j}}[AM_{\nu_{k_j}}(AN_{\nu_{k_j}} + 1) + B] + B\} &\rightarrow h', \\ \phi[AM_{\nu_{k_j}}(AN_{\nu_{k_j}} + 1) + B] &\rightarrow F(h') - \tau,\end{aligned}$$

and so

$$(2.6) \quad F(h) = F[F(h') - \tau].$$

Finally, from (2.3), (2.5) and (2.6) we get that

$$F(g) = g - \phi(A).$$

So we have proved Lemma 1.

We now prove Theorem 1.

Let $\phi \in A_G^*(E[A, B])$. Let S denote the set of limit points of $\{\phi(An + B) - \phi(n) - \phi(A) \mid n \in \mathbb{N}\}$. It is obvious that $S \neq \emptyset$. We shall prove that $S = \{0\}$.

Let $\delta \in S$. Then there exists a sequence $\{n_\nu\}_{\nu=1}^\infty$ for which

$$(2.7) \quad \phi(An_\nu + B) - \phi(n_\nu) - \phi(A) \rightarrow \delta.$$

Since G is sequentially compact, therefore we can choose a suitable convergent subsequence of $\phi(An_\nu + B)$. Let

$$(2.8) \quad \phi(An_{\nu_i} + B) \rightarrow g \quad (\in X_B).$$

From (2.7) and (2.8) we have

$$g - F(g) - \phi(A) = \delta,$$

which, using Lemma 1, implies that $\delta = 0$. Thus, we have proved that $S = \{0\}$, and so

$$\phi(An + B) - \phi(n) - \phi(A) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

This shows that $\phi \in A_G^*(\Delta[A, B])$, consequently

$$A_G^*(E[A, B]) = A_G^*(\Delta[A, B]).$$

The proof of Theorem 1 is finished.

3. PROOF OF THEOREM 2. Let $A > 0$, $B > 0$ be fixed integers.

Assume that $\phi \in A_G^*(\Delta[A, B])$, i.e. there is an element $E \in G$ such that

$$(3.1) \quad \phi(An + B) - \phi(n) - E \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Let $\chi : G \rightarrow T$ be any continuous character, where T denotes the unit circle, i.e. the set of all complex-numbers of modulus 1. Let

$$V(n) := \chi(\phi(n)) \quad \text{for all } n \in \mathbf{N}$$

and

$$C := \chi(E) \quad (E \in T).$$

Then, by (3.1), we have

$$(3.2) \quad V(An + B)(CV(n))^{-1} = \chi[\phi(An + B) - \phi(n) - E] \rightarrow \chi(0) = 1.$$

In [5] (Theorem 3) we have proved that if $V : \mathbf{N} \rightarrow T$ is a completely multiplicative function and it satisfies the relation

$$V(An + B)(CV(n))^{-1} \rightarrow 1$$

for some positive integers A, B and a non-zero complex-number C , then there exists a real-number τ such that $V(n) = n^{i\tau}$ for all $n \in \mathbf{N}$.

From (3.2) and by using this result we get immediately

$$V(n) = \chi(\phi(n)) = n^{i\tau}$$

for some real τ . Thus, by using an argument based on the proof of Theorem 1 of Z. DARÓCZY and I. KÁTAI [2], we deduce immediately that there exists a continuous homomorphism $\Psi : \mathbf{R}_x \rightarrow G$, such that $\Psi(n) = \phi(n)$ for all $n \in \mathbf{N}$.

Assume now that $A > 0$ and $B < 0$. In this case our Theorem 2 also holds, since it is easily seen that

$$A_G^*(\Delta[A, B]) \subseteq A_G^*(\Delta[A, -1])$$

and

$$A_G^*(\Delta[A, -1]) \subseteq A_G^*(\Delta[A, 1]).$$

So we have proved the first assertion of Theorem 2. The proof of the converse assertion is obvious. Thus completes the proof of Theorem 2.

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