

On compositions of distributions

By KOU HUAIZHONG (Xixing City) and BRIAN FISHER (Leicester)

Abstract. Using a double neutrix limit, the paper investigates the compositions of one-dimensional distributions and obtains a universal definition of the compositions of distributions.

1. Introduction

Since 1950, much investigation has been carried out on the compositions of distributions, see [3], [6] and [7], but only in the case where the functions involved have been continuous or at most locally summable. No meaning could be given to expressions of the form $H(\delta^{(s)}(x))$, $\delta^{(r)}(\delta^{(s)}(x))$, $[\delta^{(r)}(x)]^s$ and the like. The purpose of this paper is to achieve a new universal definition of the compositions of distributions, where a double neutrix limit is used. Accordingly, the questions not resolved are worked out.

2. Preliminaries

2.1. Neutrix and Neutrix Limit

The following definition of a neutrix was given by van der CORPUT [1]:

Definition 2.1. Let N' be a set and let N be a commutative, additive group of functions mapping N' into a commutative, additive group N'' . If N has the property that the only constant function in N is the zero function, then N is said to be a neutrix and the functions in N are said to be negligible.

Now suppose that N' is a subspace of a topological space X having a limit point y which is not contained in N' . Let N'' be the real (or complex)

numbers and let N be a commutative, additive group of functions mapping N' into N'' with the property that if N contains a function $\nu(x)$ which converges to a finite limit c as x tends to y , then $c = 0$. N is a neutrix, since if f is in N and $f(x) = c$ for all x in N' , then $f(x)$ converges to the finite limit c as x tends to y and so $c = 0$.

This leads us to a second definition of van der CORPUT [1].

Definition 2.2. Let $f(x)$ be a real (or complex) valued function defined on N' and suppose it is possible to find a constant c such that $f(x) - c$ is negligible in N . Then c is called the neutrix limit of $f(x)$ as x tends to y and we write

$$N-\lim_{x \rightarrow y} f(x) = c.$$

Note that if a neutrix limit exists, then it is unique, since if $f(x) - c$ and $f(x) - c'$ are in N , then the constant function $c - c'$ is also in N and so $c = c'$.

In the following we let N the neutrix having domain $N' = \{1, 2, \dots, n, \dots\}$, range the real numbers and $y = \infty$, with negligible functions finite linear sums of the functions

$$n^\lambda \ln^{r-1} n, \quad \ln^r n : \quad \lambda > 0, \quad r = 1, 2, \dots$$

and all functions which converge to zero in the normal sense as n tends to infinity.

We will use n or m to denote a general term in N' so that if $\{a_n\}$ is a sequence of real numbers, then $N-\lim_{n \rightarrow \infty} a_n$ means exactly the same thing as $N-\lim_{m \rightarrow \infty} a_m$.

Note that if $\{a_n\}$ is a sequence of real numbers which converges to a in the normal sense as n tends to infinity, then the sequence $\{a_n\}$ converges to a in the neutrix sense as n tends to infinity and

$$\lim_{n \rightarrow \infty} a_n = N-\lim_{n \rightarrow \infty} a_n.$$

2.2. Convolution and regularity

We now let $\rho(x)$ be any infinitely differentiable function having the following properties:

- (i) $\rho(x) = 0$ for $|x| \geq 1$,
- (ii) $\rho(x) \geq 0$,
- (iii) $\rho(x) = \rho(-x)$,

$$(iv) \int_{-1}^1 \rho(x) dx = 1.$$

Putting $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \dots$, it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$.

Now let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$. Then if f is an arbitrary distribution in $\mathcal{D}$, we define

$$f_n(x) = (f * \delta_n)(x) = \langle f(t), \delta_n(x - t) \rangle$$

for $n = 1, 2, \dots$. It follows that $\{f_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the distribution $f(x)$.

3. Compositions

Since 1983, the second author in [3], [4] and [5] has investigated the composition of distributions using neutrix limits. The following definition was given in [5] and is the most general.

Definition 3.1. Let F be a distribution in $\mathcal{D}'$ and let f be a locally summable function. We say that the distribution $F(f(x))$ exists and is equal to $h(x)$ on the interval (a, b) if

$$N\text{-}\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} F_n(f(x)) \phi(x) dx = \langle h(x), \phi(x) \rangle$$

for all test functions ϕ in $\mathcal{D}$ with support contained in the interval (a, b) , where

$$F_n(x) = (F * \delta_n)(x)$$

for $n = 1, 2, \dots$ and N is the neutrix given in the previous section.

We now give an alternative definition for the distribution $F(f)$.

Definition 3.2. Let F and f be distributions in $\mathcal{D}'$. We say that the distribution $F(f(x))$ exists and is equal to $h(x)$ on the interval (a, b) if

$$N\text{-}\lim_{n \rightarrow \infty} \left[N\text{-}\lim_{m \rightarrow \infty} \int_{-\infty}^{\infty} F_n(f_m(x)) \phi(x) dx \right] = \langle h(x), \phi(x) \rangle$$

for all ϕ in $\mathcal{D}$ with support contained in the interval (a, b) , where

$$F_n(x) = (F * \delta_n)(x), \quad f_m(x) = (f * \delta_m)(x)$$

for $m, n = 1, 2, \dots$.

In the following theorem, we show that Definition 3.2 generalizes Definition 3.1 for bounded, locally summable functions f .

Theorem 3.1. *Let F be a distribution in $\mathcal{D}$ and let f be a bounded, locally summable function. If the distribution $F(f(x))$ exists and equals $h(x)$ under Definition 3.1 on the interval (a, b) , then $F(f(x))$ also exists under Definition 3.2 on the interval (a, b) and is equal to $h(x)$.*

PROOF. Suppose that $F(f(x))$ exists and equals $h(x)$ under Definition 3.1 on the interval (a, b) . Then

$$N\text{-}\lim_{n \rightarrow \infty} \langle F_n(f(x)), \phi(x) \rangle = \langle h(x), \phi(x) \rangle$$

for all ϕ in $\mathcal{D}(a, b)$. Now

$$\lim_{m \rightarrow \infty} \int_I |f_m(x) - f(x)| dx = 0,$$

for any bounded interval I , since f is a bounded, summable function. Further, since F_n is a continuously differentiable function and f, f_n are bounded, it follows that

$$|F_n(f_m(x)) - F_n(f(x))| \leq K_n |f_m(x) - f(x)|,$$

for some K_n . We therefore have

$$\left| \int_a^b [F_n(f_m(x)) - F_n(f(x))] \phi(x) dx \right| \leq MK_n \int_c^d |f_m(x) - f(x)| dx,$$

where

$$M = \sup\{|\phi(x)|\}$$

and $[c, d]$ is a bounded interval containing the support of ϕ and so

$$\lim_{m \rightarrow \infty} \int_a^b F_n(f_m(x)) \phi(x) dx = \int_a^b F_n(f(x)) dx,$$

or equivalently

$$N\text{-}\lim_{m \rightarrow \infty} \langle F_n(f_m(x)), \phi(x) \rangle = \langle F_n(f(x)), \phi(x) \rangle.$$

Thus

$$\begin{aligned} N\text{-}\lim_{n \rightarrow \infty} \left[N\text{-}\lim_{m \rightarrow \infty} \langle F_n(f_m(x)), \phi(x) \rangle \right] &= N\text{-}\lim_{n \rightarrow \infty} \langle F_n(f(x)), \phi(x) \rangle \\ &= \langle F(f(x)), \phi(x) \rangle \\ &= \langle h(x), \phi(x) \rangle, \end{aligned}$$

and it follows that $F(f(x))$ exists and equals $h(x)$ by Definition 3.2.

It is an open question as to whether Definition 3.2 is a generalization of Definition 3.1 for all locally summable functions.

From now on, the compositions that we will consider will be using Definition 3.2.

Theorem 3.2. *Let F be a bounded, continuous, summable function on the real line. Then the distribution $F(\delta^{(s)}(x))$ exists on the real line and*

$$F(\delta^{(s)}(x)) = F(0),$$

for $s = 0, 1, 2, \dots$.

PROOF. We put

$$F_n(x) = (F * \delta_n)(x), \quad \delta_m^{(s)}(x) = (\delta^{(s)} * \delta_m)(x)$$

for $m, n = 1, 2, \dots$.

Choosing an arbitrary $\varepsilon > 0$, there exists an M such that $m\varepsilon > 1$ for $m > M$. Then with $m > M$, we have

$$\delta_m^{(s)}(x) = m^{s+1} \rho(mx) = 0$$

for $|x| > \varepsilon$ and so

$$F_n(\delta_m^{(s)}(x)) = F_n(0)$$

for $|x| > \varepsilon$. Thus, for arbitrary ϕ in $\mathcal{D}$ with support contained in the interval (a, b) , which we may suppose contains the origin,

$$\begin{aligned} \langle F_n(\delta_m^{(s)}(x)), \phi(x) \rangle &= \int_a^b F_n(\delta_m^{(s)}(x)) \phi(x) dx \\ &= F_n(0) \int_a^{-\varepsilon} \phi(x) dx + F_n(0) \int_{\varepsilon}^b \phi(x) dx + \int_{-\varepsilon}^{\varepsilon} F_n(\delta_m^{(s)}(x)) \phi(x) dx, \end{aligned}$$

for $m > M$. Thus

$$\begin{aligned} |\langle F_n(\delta_m^{(s)}(x)) - F_n(0), \phi(x) \rangle| &= \left| \int_{-\varepsilon}^{\varepsilon} [F_n(\delta_m^{(s)}(x)) - F_n(0)] \phi(x) dx \right| \\ &\leq 2\varepsilon K \end{aligned}$$

for $m > M$, where

$$K = \sup \{ |[F_n(\delta_m^{(s)}(x)) - F_n(0)] \phi(x)| : m, n = 1, 2, \dots; x \in \mathbf{R} \} < \infty,$$

since F_n and ϕ are bounded functions. It follows that

$$\lim_{m \rightarrow \infty} \langle F_n(\delta_m^{(s)}(x)), \phi(x) \rangle = \langle F_n(0), \phi(x) \rangle$$

and so

$$\begin{aligned} N\text{-}\lim_{n \rightarrow \infty} \left[N\text{-}\lim_{m \rightarrow \infty} \langle F_n(\delta_m^{(s)}(x)), \phi(x) \rangle \right] &= \lim_{n \rightarrow \infty} \langle F_n(0), \phi(x) \rangle \\ &= \langle F(0), \phi(x) \rangle. \end{aligned}$$

This proves that $F(\delta^{(s)}(x))$ exists and is equal to $F(0)$ for $s = 0, 1, 2, \dots$.

Theorem 3.3. *The distribution $H(\delta^{(s)}(x))$ exists on the real line and*

$$H(\delta^{(s)}(x)) = \frac{1}{2},$$

for $s = 0, 1, 2, \dots$, where H denotes Heaviside's function.

PROOF. We put

$$H_n(x) = (H * \delta_n)(x)$$

for $n = 1, 2, \dots$, so that

$$\begin{aligned} H_n(x) &= \begin{cases} 1, & x > 1/n, \\ \int_{-1/n}^x \delta_n(t) dt, & |x| \leq 1/n, \\ 0, & x < -1/n, \end{cases} \\ 0 \leq H_n(x) &\leq 1, \quad H_n(0) = \frac{1}{2}, \end{aligned}$$

for $n = 1, 2, \dots$.

Choosing arbitrary $\varepsilon > 0$, there exists M such that $m\varepsilon > 1$ for $m > M$. It then follows as above that

$$H_n(\delta_m^{(s)}(x)) = H_n(0) = \frac{1}{2},$$

for $|x| > \varepsilon$ and $m > M$ and so

$$\left| \left\langle H_n(\delta_m^{(s)}(x)) - \frac{1}{2}, \phi(x) \right\rangle \right| = \left| \int_{-\varepsilon}^{\varepsilon} \left[H_n(\delta_m^{(s)}(x)) - \frac{1}{2} \right] \phi(x) dx \right| \leq \int_{-\varepsilon}^{\varepsilon} |\phi(x)| dx$$

for $m > M$ and arbitrary ϕ in $\mathcal{D}$. The result of the theorem follows as above.

Theorem 3.4. *Let F be a bounded, summable function on the real line which is continuous everywhere except for a simple discontinuity at the origin. Then the distribution $F(\delta^{(s)}(x))$ exists on the real line and*

$$F(\delta^{(s)}(x)) = \frac{1}{2}[F(0+) + F(0-)]$$

for $s = 0, 1, 2, \dots$.

PROOF. Let $F(0+) - F(0-) = c$. Then the function G defined by

$$G(x) = F(x) - cH(x)$$

satisfies the conditions of Theorem 3.2. Thus

$$G(\delta^{(s)}(x)) = G(0) = F(0-),$$

and so

$$\begin{aligned} G(\delta^{(s)}(x)) + cH(\delta^{(s)}(x)) &= F(0-) + \frac{1}{2}[F(0+) - F(0-)] \\ &= \frac{1}{2}[F(0+) + F(0-)] \end{aligned}$$

for $s = 0, 1, 2, \dots$. The result of the theorem follows.

Theorem 3.5. *The distribution $\delta^{(r)}(\delta^{(s)}(x))$ exists on the real line and*

$$\delta^{(r)}(\delta^{(s)}(x)) = 0$$

for $r, s = 0, 1, 2, \dots$.

PROOF. Choosing arbitrary $\varepsilon > 0$, there exists M such that $m\varepsilon > 1$ for $m > M$. Then $m\varepsilon > 1$ and $|x| > \varepsilon$ implies that $\rho^{(s)}(mx) = 0$ and so

$$\begin{aligned} \delta_n^{(r)}(\delta_m^{(s)}(x)) &= n^{r+1} \rho^{(r)}(nm^{s+1} \rho^{(s)}(mx)) \\ &= n^{r+1} \rho^{(r)}(0) \end{aligned}$$

for $|x| > \varepsilon$ and $m > M$.

Thus, for $m > M$ and all ϕ in $\mathcal{D}$, we have

$$\begin{aligned} \langle \delta_n^{(r)}(\delta_m^{(s)}(x)), \phi(x) \rangle &= n^{r+1} \rho^{(r)}(0) \int_{|x| > 1/m} \phi(x) dx + \\ &+ n^{r+1} \int_{|x| < 1/m} \rho^{(r)}(nm^{s+1}(mx)) \phi(x) dx \rightarrow n^{r+1} \rho^{(r)}(0) \int_{-\infty}^{\infty} \phi(x) dx \end{aligned}$$

as m tends to infinity. It follows that

$$\lim_{m \rightarrow \infty} \langle \delta_n^{(r)}(\delta_m^{(s)}(x)), \phi(x) \rangle = n^{r+1} \langle \rho^{(r)}(0), \phi(x) \rangle$$

and so

$$\begin{aligned} N\text{-}\lim_{n \rightarrow \infty} \left[N\text{-}\lim_{m \rightarrow \infty} \langle \delta_n^{(r)}(\delta_m^{(s)}(x)), \phi(x) \rangle \right] &= N\text{-}\lim_{n \rightarrow \infty} n^{r+1} \langle \rho^{(r)}(0), \phi(x) \rangle \\ &= 0. \end{aligned}$$

The result of the theorem follows. This completes the proof of the theorem.

Theorem 3.6. *The distribution $[\delta^{(r)}(x)]^s$ exists on the real line and*

$$(1) \quad [\delta^{(r)}(x)]^s = \frac{(-1)^{rs+s-1} c(\rho, r, s)}{(rs+s-1)!} \delta^{(rs+s-1)}(x)$$

for $r = 0, 1, 2, \dots$ and $s = 2, 3, \dots$, where

$$c(\rho, r, s) = \int_{-1}^1 [\rho^{(r)}(y)]^s y^{rs+s-1} dy.$$

In particular

$$(2) \quad [\delta^{(r)}(x)]^s = 0$$

for even s .

PROOF. Put

$$\begin{aligned} (x^s)_n &= x^s * \delta_n(x) = \int_{-1/n}^{1/n} (x-t)^s \delta_n(t) dt \\ &= \sum_{i=0}^s (-1)^{s-i} \binom{s}{i} x^i \int_{-1/n}^{1/n} t^{s-i} \delta_n(t) dt \end{aligned}$$

for $n = 1, 2, \dots$, where

$$\binom{s}{i} = \frac{s!}{i!(s-i)!}.$$

Then

$$(3) \quad [(\delta_m^{(r)}(x))^s]_n = \sum_{i=0}^s (-1)^{s-i} \binom{s}{i} [\delta_m^{(r)}(x)]^i \int_{-1/n}^{1/n} t^{s-i} \delta_n(t) dt,$$

where

$$\delta_m^{(r)}(x) = m^{r+1} \rho^{(r)}(mx),$$

the support of $\delta_m^{(r)}$ being contained in the interval $[-1/m, 1/m]$. Making the substitution $y = mx$ we have

$$\int_{-1/m}^{1/m} [\delta_m^{(r)}(x)]^i x^j dx = m^{ri+i-j-1} \int_{-1}^1 [\rho^{(r)}(y)]^i y^j dy.$$

It follows that

$$(4) \quad N\text{-}\lim_{m \rightarrow \infty} \int_{-1/m}^{1/m} [\delta_m^{(r)}(x)]^i x^j dx = 0,$$

for $i = 0, 1, \dots, s$, $j = 0, 1, 2, \dots$ and $j \neq ri + i - 1$.

In the particular case $j = ri + i - 1$ we have

$$(5) \quad \int_{-\infty}^{\infty} [\delta_m^{(r)}(x)]^i x^{ri+i-1} dx = \int_{-1}^1 [\rho^{(r)}(y)]^i y^{ri+i-1} dy = c(\rho, r, i).$$

Now let ϕ be an arbitrary function in $\mathcal{D}$. Then by Taylor's theorem we have

$$\phi(x) = \sum_{j=0}^{rs+s-1} \frac{\phi^{(j)}(0)}{j!} x^j + \frac{\phi^{(rs+s)}(\xi x)}{(rs+s)!} x^{rs+s},$$

where $0 \leq \xi \leq 1$. Thus

$$\begin{aligned} \int_{-1/m}^{1/m} [(\delta_m^{(r)}(x))^i]_n \phi(x) dx &= \sum_{j=0}^{rs+s-1} \frac{\phi^{(j)}(0)}{j!} \int_{-1/m}^{1/m} [\delta_m^{(r)}(x)]^i x^j dx \\ &\quad + \int_{-1/m}^{1/m} \frac{\phi^{(rs+s)}(\xi x)}{(rs+s)!} [\delta_m^{(r)}(x)]^i x^{rs+s} dx, \end{aligned}$$

where

$$\begin{aligned} \left| \int_{-1/m}^{1/m} \frac{\phi^{(rs+s)}(\xi x)}{(rs+s)!} [\delta_m^{(r)}(x)]^i x^{rs+s} dx \right| &\leq \\ &\leq \frac{2m^{(r+1)(i-s)-1}}{(rs+s)!} \sup\{|\phi^{(rs+s)}(x)|\} \cdot \sup\{|\rho^{(r)}(x)|\} \\ &\rightarrow 0 \end{aligned}$$

as m tends to infinity for $i = 0, 1, \dots, s$.

Using equations (4) and (5), it follows that

$$N\text{-}\lim_{m \rightarrow \infty} \int_{-1/m}^{1/m} [(\delta_m^{(r)})^i]_n \phi(x) dx = \begin{cases} 0, & i = 0, \\ \frac{c(\rho, r, i) \phi^{(ri+i-1)}(0)}{(ri+i-1)!}, & i = 1, \dots, s. \end{cases}$$

It now follows from equation (3) that

$$\begin{aligned} \langle [(\delta_m^{(r)}(x))^s]_n, \phi(x) \rangle &= \\ &= \sum_{i=0}^s (-1)^i \binom{s}{i} \int_{-1/m}^{1/m} [(\delta_m^{(r)}(x))^{s-i}]_n \phi(x) dx \cdot \int_{-1/n}^{1/n} t^{s-i} \delta_n(t) dt \end{aligned}$$

and it follows from what we have just proved that

$$\begin{aligned} N\text{-}\lim_{m \rightarrow \infty} \langle [(\delta_m^{(r)}(x))^s]_n, \phi(x) \rangle &= \\ &= \sum_{i=1}^s (-1)^{s-i} \binom{s}{i} \frac{c(\rho, r, i) \phi^{(ri+i-1)}(0)}{(ri+i-1)!} \int_{-1/n}^{1/n} t^{s-i} \delta_n(t) dt, \end{aligned}$$

where

$$\int_{-1/n}^{1/n} t^{s-i} \delta_n(t) dt = \begin{cases} n^{i-s} \int_{-1}^1 u^{s-i} \rho(u) du, & i = 1, \dots, s-1 \\ 1, & i = s, \end{cases}$$

Thus

$$\begin{aligned} N-\lim_{n \rightarrow \infty} \left[N-\lim_{m \rightarrow \infty} \langle [(\delta_m^{(r)}(x))^s]_n, \phi(x) \rangle \right] &= \frac{c(\rho, r, s) \phi^{(rs+s-1)}(0)}{(rs+s-1)!} \\ &= \frac{(-1)^{rs+s-1} c(\rho, r, s)}{(rs+s-1)!} \langle \delta^{(rs+s-1)}(x), \phi(x) \rangle \end{aligned}$$

and equation (1) follows. Equation (2) follows on noticing that

$$[\rho^{(r)}(y)]^s y^{rs+s-1}$$

is an odd function for even s and so $c(\rho, r, s) = 0$ for even s . This completes the proof of the theorem.

The next definition for the product of two distributions was given in [2].

Definition 3.3. Let f and g be distributions in D' and let

$$f_n(x) = (f * \delta_n)(x), \quad g_n(x) = (g * \delta_n)(x).$$

Then the product $f.g$ is defined to exist and be equal to the distribution h on the interval (a, b) if

$$N-\lim_{n \rightarrow \infty} \langle f_n(x) g_n(x), \phi(x) \rangle = \langle h(x), \phi(x) \rangle$$

for all test functions ϕ in $\mathcal{D}$ with support contained in the interval (a, b) .

We note that with this definition of the product of two distributions, the definition of the distribution f^2 as the composition of the function x^2 and the distribution f if it exists, is distinct from the definition of the product $f.f$ if it exists. However, the following theorem holds:

Theorem 3.7. *Let f be a distribution in D' . Then the distribution f^2 exists on the interval (a, b) if and only if the distribution $f.f$ exists on the interval (a, b) and then*

$$f^2 = f \cdot f$$

on the interval (a, b) .

PROOF. It follows as in the proof of Theorem 3.6 that

$$[(f_m(x))^2]_n = \int_{-1/n}^{1/n} t^2 \delta_n(t) dt - 2f_m(x) \int_{-1/n}^{1/n} t \delta_n(t) dt + [f_m(x)]^2 \int_{-1/n}^{1/n} \delta_n(t) dt,$$

where

$$\lim_{n \rightarrow \infty} \int_{-1/n}^{1/n} t^2 \delta_n(t) dt = \lim_{n \rightarrow \infty} \int_{-1/n}^{1/n} t \delta_n(t) dt = 0, \quad \int_{-1/n}^{1/n} \delta_n(t) dt = 1.$$

Then it follows that f^2 exists on the interval (a, b) , if and only if

$$N\text{-}\lim_{n \rightarrow \infty} \left[N\text{-}\lim_{m \rightarrow \infty} \langle [f_m(x)]^2 \rangle_n, \phi(x) \right]$$

exists and is equal to

$$N\text{-}\lim_{m \rightarrow \infty} \langle [f_m(x)]^2 \rangle_n, \phi(x) = N\text{-}\lim_{n \rightarrow \infty} \langle f_n(x) f_n(x), \phi(x) \rangle,$$

for all ϕ in $\mathcal{D}$ with support contained in the interval (a, b) . That is, if and only if $f.f$ exists on the interval (a, b) .

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KOU HUAI ZHONG
DEPARTMENT OF MATHEMATICS
HENAN NORMAL UNIVERSITY
XIXING CITY, HENAN PROVINCE
PEOPLE'S REPUBLIC OF CHINA

BRIAN FISHER
DEPARTMENT OF MATHEMATICS
THE UNIVERSITY, LEICESTER
LE1 7RH
ENGLAND

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