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On fixed points in noncomplete metric spaces

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A common fixed point theorem of a pair of mappings of a metric space into itself is proved, generalizing the results of K. Iseki [3], R. Kannan [4] and S. P. Singh [5].

1. In [1] D. Delbosco gives a unified approach for contractive mappings considering the set $\mathcal{G}$ of all continuous functions $g : [0, \infty)^3 \to [0, \infty)$ satisfying the conditions:

(i) $g(1, 1, 1) = h < 1$,

(ii) if $u, v \in [0, \infty)$ are such that $u \le g(v, v, u)$ or $u \le g(u, v, v)$ or $u \le g(v, u, v)$ then $u \le hv$,

and proving the following

Theorem A. *Let S and T be two mappings of a complete metric space (X, d) into itself satisfying the inequality*

$$d(Sx, Ty) \le g(d(x, y), d(x, Sx), d(y, Ty))$$

for all $x, y \in X$, where g is in $\mathcal{G}$. Then S and T have a unique common fixed point.

Some fixed point theorems for mappings on noncomplete metric space were proved by several authors: R. Kannan [4], S. P. Singh [5], M. Tasković [6].

The aim of this note is to prove a similar result to Delbosco's result for mappings defined on a noncomplete metric space (X, d) into itself, generalizing the results of R. Kannan [4] and S. P. Singh [5].

2. We consider the set $\mathcal{L}$ of all continuous functions $g : [0, \infty)^3 \to [0, \infty)$ with the property that if $u, v \in [0, \infty)$ are such that $u < g(v, v, u)$ or $u < g(v, u, v)$ or $u < g(u, v, v)$ then $u < v$.

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Theorem 1. *Let S and T be two continuous mappings of a metric space into itself satisfying the inequality*

$$d(Sx,Ty) < g(d(x,y),d(x,Sx),d(y,Ty)) \tag{1}$$

for all $(x,y) \in X \times X \setminus \{(x,x) \mid x \in X \text{and } Sx = Tx\}$ where g is in $\mathcal{L}$. If there is a $x_0 \in X$ such that the sequence $\{(Ts)^n x_0\}$ has a subsequence $\{(TS)^{n_i} x_0\}$ converging to a point $x \in X$, we have that x is the unique common fixed point of S and T.

Proof. We consider the sequence $\{x_n\}$ defined by

$$x_{2n+1} = S(TS)^n x_0, \qquad x_{2n} = (TS)^n x_0, \quad n \geq 0.$$

We observe that if there is a $z \in X$ such that $Sz = z$ then $Tz = z$. If $Tz \neq z$ we would obtain that

$$d(z,Tz) = d(Sz,Tz) < g(0,0,d(z,Tz))$$

and since $g \in \mathcal{L}$ we deduce that $d(z,Tz) < 0$, contradiction. Analogous we can prove that if there is a $z \in X$ such that $Tz = z$ then $Sz = z$.

If there is a $z \in X$ such that $Sz = Tz = z$ we have that there is no other point $y \in X$ such that $Sy = Ty = y$ since otherwise we would have that

$$d(y,z) = d(Sy,Tz) < g(d(y,z),0,0)$$

and since $g \in \mathcal{L}$ we deduce that $d(y,z) < 0$, contradiction.

If there is a $n \in \mathbb{N}$ such that $x_{2n+1} = x_{2n}$ or $x_{2n+1} = x_{2n+2}$ by the preceding remarks we deduce that $(TS)^n x_0 = x$ (respectively $S(TS)^n\ x_0 = x$) and $Sx = Tx = x$. More then that, x is the unique common fixed point of S and T.

We suppose that for every $n \in \mathbb{N}$ we have $x_{2n+1} \neq x_{2n}$ and $x_{2n+1} \neq x_{2n+2}$ and we deduce that

$$\begin{aligned}
d(x_{2n+1},x_{2n+2}) &= d(S(TS)^n x_0,(TS)^{n+1}x_0) < \\
&< g(d((TS)^n x_0,S(TS)^n x_0),d((TS)^n x_0,S(TS)^n x_0), \\
&\qquad d(S(TS)^n x_0,(TS)^{n+1}x_0)) = \\
&= g(d(x_{2n},x_{2n+1}),d(x_{2n},x_{2n+1}),d(x_{2n+1},x_{2n+2})), \quad n \geq 0\,.
\end{aligned}$$

Since $g \in \mathcal{L}$ we have that $d(x_{2n+1},x_{2n+2}) < d(x_{2n},x_{2n+1})$.

Analogous we have

$$\begin{aligned} d(x_{2n}, x_{2n+1}) &= d((TS)^n x_0, S(TS)^n x_0) < \\ &< g(d((TS)^n x_0, S(TS)^{n-1} x_0), d((TS)^n x_0, S(TS)^n x_0), \\ &\qquad d((TS)^n x_0, S(TS)^{n-1} x_0)) = \\ &= g(d(x_{2n}, x_{2n-1}), d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n-1})) \end{aligned}$$

and since $g \in \mathcal{L}$ we deduce that $d(x_{2n}, x_{2n+1}) < d(x_{2n}, x_{2n-1})$.

We have so proved that the sequence $\{d(x_k, x_{k+1})\}$ is monotone decreasing. We deduce that

$$\begin{aligned} d(x_{2n_i}, x_{2n_i+1}) = d((TS)^{n_i} x_0, S(TS)^{n_i} x_0) > d((TS^{n_i+1} x_0, S(TS)^{n_i} x_0) > \\ > \cdots > d((TS)^{n_{i+1}} x_0, S(TS)^{n_{i+1}} x_0), \quad i \geq 1 . \end{aligned}$$

Since $T,\ S$ are continuous, letting $n_i \to \infty$, we deduce that

$$d(x, Sx) = d(TSx, Sx) . \tag{2}$$

If $Sx = x$ we deduce that $Tx = Sx = x$ and x is the unique common fixed point of T and S.

Let us suppose that $Sx \neq x$. We have that

$$d(TSx, Sx) < g(d(Sx, x), d(TSx, Sx), d(x, Sx))$$

and since $g \in \mathcal{L}$ we deduce that $d(TSx, Sx) < d(Sx, x)$ which contradicts relation (2).

Corollary 1. (S. P. Singh [5]). *Let T be a continuous mapping of a metric space into itself satisfying the inequality*

$$d(Tx, Ty) < \frac{1}{2}(d(x, Tx) + d(y, Ty)) \tag{3}$$

for all $x \neq y$. If there is a $x_0 \in X$ such that sequence $\{T^n x_0\}$ has a subsequence $\{T^{n_i} x_0\}$ converging to a point $x \in X$ then x is the unique fixed point of T.

Proof. We take $g : [0, \infty)^3 \to [0, \infty),\ g(x_1, x_2, x_3) = \frac{1}{2}(x_2 + x_3)$ and $T = S$ in Theorem 1.

Corollary 2. *Let T be a continuous mapping of a metric space into itself satisfying the inequality*

$$d(Tx,Ty) < g(d(x,y), d(x,Tx), d(y,Ty)) \tag{4}$$

for all $x \neq y$ where $g \in \mathcal{L}$. If there is a $x_0 \in X$ such that the sequence $\{T^n x_0\}$ has a subsequence $\{T^{n_i} x_0\}$ converging to a point $x \in X$ then x is the unique fixed point of T.

PROOF. We take $T = S$ in Theorem 1.

Corollary 3. (K. ISEKI [3]). *Let (X,d) be a metric space and S a continuous mapping of X into itself satisfying*

$$d(Sx,Sy) \leq \alpha d(x,y)$$

for all $x,y \in X$ where $0 < \alpha < 1$. If for some $x_0 \in X$, the sequence $x_1 = Sx_0,\ x_2 = Sx_1,\ x_3 = Sx_2, \dots$ contains s convergent subsequence which converges to some $x \in X$, then x is the unique fixed point of S.

PROOF. If the sequence $\{S^{2n}x_0\}$ contains a subsequence converging to x we can apply our theorem with $g : [0,\infty)^3 \to [0,\infty)$,
$g(x_1,x_2,x_3) = \alpha x_1$.

Let us suppose that the sequence $\{S^{2n+1}x_0\}$ contains a subsequence converging to x. Since $d(S^{n+1}x_0, S^n x_0) \leq \alpha^n d(Sx_0, x_0)$ we deduce that the sequence $\{S^{2n}x_0\}$ contains a subsequence converging to x and so we obtain the result.

3. We prove now, by mean of an example, that Corollary 2 is stronger than the result of S. P. SINGH [5]:

Example 1. Consider the mapping $T : [0,1) \to [0, \frac{1}{3}),\ Tx = \frac{x}{3}$ and let d be the euclidean metric. We have that

$$d(Tx,Ty) = \frac{|x-y|}{3}$$
$$\frac{1}{2}(d(x,Tx) + d(y,Ty)) = \frac{x+y}{3}$$

and for $y = 0 < x$ we do not have that

$$d(Tx,Ty) < \frac{1}{2}(d(x,Tx) + d(y,Ty))$$

so that we can not apply Corollary 1.

If we consider the mapping $g : [0,\infty)^3 \to [0,\infty),\ g(x_1,x_2,x_3) = \frac{1}{4}(x_1+x_2+x_3)$ we have that $g \in \mathcal{L}$ and

$$d(Tx,Ty) < g(d(x,y),d(x,Tx),d(y,Ty))$$

if $x \neq y$ since

$$\frac{|x-y|}{3} < \frac{1}{4}\left(|x-y| + \frac{2x}{3} + \frac{2y}{3}\right)$$

if $(x,y) \neq (0,0)$. We can so apply Corollary 2 to this function.

4. To compare Theorem 1 with the theorem of DELBOSCO we see that we have omitted the completeness of the metric space (X,d) and instead we have assumed other conditions on the mappings S and T. These conditions do not guarantae the completeness of the space:

Example 2. Let $X = [0,1]\cap\mathbb{Q},\ g : [0,\infty)^3 \to [0,\infty),\ g(x_1,x_2,x_3) = \alpha(x_2+x_3)$ with $\frac{1}{3} < \alpha < \frac{1}{2}$ and $T,S : X \to X,\ Tx = \frac{x}{4},\ Sx = \frac{x}{5}$ and let d be the euclidean metric. Both S,T are continuous and since $TSx = \frac{x}{20}$ we can take $x_0 = 0$ and then the existence of a convergent subsequence of the sequence $\{(TS)^n x_0\}$ is evident. We have also that

$$d(Sx,Ty) = \left|\frac{x}{5} - \frac{y}{4}\right|$$

$$Sx = Tx \ \text{ if and only if } \ x = 0\,,$$

$$g(d(x,y),d(x,Sx),d(y,Ty)) = \alpha(d(x,Sx)+d(y,Ty)) = \alpha\left(\frac{4x}{5} + \frac{3y}{4}\right)$$

and because $\alpha > \frac{1}{3}$ we have that

$$\alpha\left(\frac{4x}{5}+\frac{3y}{4}\right) = \alpha\left(\frac{16x+15y}{20}\right) > \frac{4x+5y}{20} \geq \frac{|5y-4x|}{20}$$

if $(x,y) \in X \times X \setminus \{(0,0)\}$.

5. *Examples 3.* Let us consider $X = [0,1)$ with the euclidean metric and let $S : [0,1) \to [0,1),\ Sx = \frac{x}{2}$ for $x \in (0,1)$ and $S(0) = \frac{1}{2}$. We have that the function $g : [0,\infty)^3 \to [0,\infty),\ g(x_1,x_2,x_3) = \max\{x_1,x_2,x_3\}$ is in $\mathcal{L}$ and

$$d(Sx,Sy) < g(d(x,y),d(x,Sx),d(y,Sy))$$

for $(x,y) \in X \times X \setminus \{(x,x),\ x \in X\}$ since if $x \neq 0,\ y \neq 0,\ x \neq y$, then

$$d(Sx,Sy) = \frac{|x-y|}{2} < |x-y| \leq g\left(|x-y|, \frac{x}{2}, \frac{y}{2}\right)$$

and if $x = 0,\ y \neq 0$ then

$$d(Sx, Sy) = \left| \frac{1}{2} - \frac{y}{2} \right| < \frac{1}{2} \leq g\left(y, \frac{1}{2}, \frac{y}{2} \right) .$$

The function S has no fixed point although it satisfies condition (1) and $S^{2n}(0) \to 0$ as $n \to \infty$. This shows that the result may be not true if we drop the hypothesis that S, T are continuous.

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