

A note on the strong Schwarz inequality in Hilbert A -modules

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Abstract. The aim of this note is to show that one of the axioms of Hilbert A -modules due to SAWOROTNOW is redundant and, at the same time, to give a direct proof of the so-called strong Schwarz inequality which avoids the explicit use of the spectral representation of a normal element of a proper H^* -algebra.

Throughout this note A denotes a proper H^* -algebra, i.e. A is a Banach algebra whose norm is a Hilbert space norm and which has an involution $*$: $x \mapsto x^*$ such that $(x, yz^*) = (xz, y) = (z, x^*y)$ for all $x, y, z \in A$. A projection in A is a nonzero element e of A for which $e^2 = e = e^*$; e is called primitive if it cannot be represented as a sum of two mutually orthogonal projections of A . In the sequel $\{e_\alpha : \alpha \in \Lambda\}$ stands for a fixed maximal family of mutually orthogonal projections. An element $a \in A$ is called positive ($a \geq 0$) if $(ax, x) \geq 0$ holds whenever $x \in A$. For every $a \in A$ there exists a unique positive element $|a|$ of A such that $|a|^2 = a^*a$. If $a \in A$, then let L_a be the bounded linear operator defined by $L_ax = ax$ ($x \in A$) and denote $C(A)$ the closure of $\{L_a : a \in A\}$ in the norm of the B^* -algebra of bounded linear operators on A .

By the trace-class of A we mean the ideal $\tau(A) = \{xy : x, y \in A\}$. There is a positive linear functional tr (called trace) on $\tau(A)$ for which $tr xy^* = tr y^*x = (x, y)$, $\overline{tr a} = tr a^*$ whenever $x, y \in A$, $a \in \tau(A)$. Then one can define a norm τ on $\tau(A)$ by letting $\tau(a) = tr|a|$ ($a \in \tau(A)$).

As for the detailed discussion of H^* -algebras and their trace-classes we refer to [1], [3] and [4].

Definition. Let H be a (right) A -module. Suppose that $[\cdot, \cdot] : H \times H \rightarrow \tau(A)$ is a function with the following properties:

- (1) $[f, g + h] = [f, g] + [f, h];$
- (2) $[f, ga] = [f, g]a;$
- (3) $[f, g]^* = [g, f];$
- (4) $[f, f] \geq 0$

for every $f, g, h \in H$ and $a \in A$. Then $[\cdot, \cdot]$ is called a $(\tau(A)$ -valued) generalized semi-inner product on H .

In the proof of our theorem we need the following

Lemma. *If $x \in A$, then*

$$|([f, g]x, x)|^2 \leq ([f, f]x, x) ([g, g]x, x)$$

holds for every $f, g \in H$.

PROOF. Let $f, g \in H$. Suppose that $e \in A$ is a projection. Then one can easily verify that

$$\begin{aligned} 0 &\leq ([f + g(\lambda e), f + g(\lambda e)]e, e) \\ &= ([f, f]e, e) + 2 \operatorname{Re} \lambda ([f, g]e, e) + |\lambda|^2 ([g, g]e, e) \end{aligned}$$

for each $\lambda \in \mathbf{C}$. It implies that

$$|([f, g]e, e)|^2 \leq ([f, f]e, e) ([g, g]e, e).$$

Substitute now f, g and e by fx, gx and the sum of a finite subset of $\{e_\alpha : \alpha \in \Lambda\}$, respectively. Then, by [1, Theorem 4.1], we have the desired inequality.

Remark. We note that the proof of this lemma would be much simpler if there were an adequate vector space structure on H .

Remark also that the inequality in the Lemma directly implies the validity of the so-called weak Schwarz inequality [2, Axiom 5 in Definition 1].

Now we are in a position to prove the following

Theorem. *If $f, g \in H$, then $(\tau[f, g])^2 \leq \tau[f, f]\tau[g, g]$.*

PROOF. Let $a \in A$. Suppose that F is a finite subset of Λ . If $e = \sum\{e_\alpha : \alpha \in F\}$, then, by the Lemma, we have

$$\begin{aligned} \left| \sum (a[f, g]e_\alpha, e_\alpha) \right|^2 &= |a[f, g]e, e|^2 \leq \\ &\leq ([f, f]a^*e, a^*e)([g, g]e, e) \leq \operatorname{tr} a[f, f]a^* \tau[g, g] = \\ &= \operatorname{tr} a^*a[f, f]\tau[g, g], \end{aligned}$$

where the involved sum is taken over F . By [3, Corollary 2 and Lemma 5], the last expression is not greater than $\|L_a\|^2 \tau[f, f] \tau[g, g]$ which implies that

$$|\operatorname{tr} S[f, g]|^2 \leq \|S\|^2 \tau[f, f] \tau[g, g]$$

holds for every $S \in C(A)$. The inequality follows from [4, Theorem 1].

Remark. Using a similar argument it is easy to show that

$$[f, g]^* [f, g] \leq (\tau[f, f]) [g, g]$$

($f, g \in H$). In fact, let $x \in A$. Then, by the Lemma, we have

$$\begin{aligned} ([f[f, g], g]x, x)^2 &\leq ([f, f][f, g]x, [f, g]x) ([g, g]x, x) \leq \\ &\leq \| [f, f]^{\frac{1}{2}} \|^2 \| [f, g]x \|^2 ([g, g]x, x) = \tau[f, f] ([f[f, g], g]x, x) ([g, g]x, x) \end{aligned}$$

where $[f, f]^{\frac{1}{2}}$ denotes the unique positive element of A whose square is $[f, f]$. Now the statement is obvious.

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