

## On the divisibility of matrices associated with multiplicative functions

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**Abstract.** Suppose that  $n$  and  $k$  are positive integers. Let  $S = \{x_1, \dots, x_n\}$  be a sequence of  $n$  distinct positive integers, and let  $f$  be an integer-valued multiplicative function. The sequence  $S$  is called a divisor chain if  $x_{\sigma(1)} | \dots | x_{\sigma(n)}$  for some permutation  $\sigma$  on  $\{1, \dots, n\}$ . We say that the sequence  $S$  consists of  $k$  coprime divisor chains if  $S$  can be partitioned as the union of  $k$  divisor chains  $S_1, \dots, S_k$  such that each element of  $S_i$  is coprime to each element of  $S_j$  for all integers  $i$  and  $j$  with  $1 \leq i \neq j \leq k$ . In this paper, we show that for any divisor chain  $S$ , the matrix  $(f(S))$  with entries  $f(\gcd(x_i, x_j))$  divides the matrix  $(f[S])$  with entries  $f(\text{lcm}(x_i, x_j))$  in the ring  $M_n(\mathbf{Z})$  of  $n \times n$  matrices over the integers if and only if  $f(\min(S)) | f(x_i)$  for all integers  $i \in \{1, \dots, n\}$ . This strengthens a result of Hong [14]. For any positive integer  $a$  and any sequence  $S$  consisting of two coprime divisor chains with  $1 \notin S$ , we show that the matrix  $(f(S^a))$  divides the matrix  $(f[S^a])$  in  $M_n(\mathbf{Z})$ , where  $S^a := \{x_1^a, \dots, x_n^a\}$ . This confirms a conjecture of Chen and Hong. We show also that such factorization is no longer true in general if  $S$  consists of at least three coprime divisor chains with  $1 \notin S$ . We conjecture that if  $k \geq 3$ , then the GCD matrix  $(S)$  does not divide the LCM matrix  $[S]$  in the ring  $M_k(\mathbf{Z})$  if  $S$  consists of the first  $k$  odd prime numbers.

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