

Einstein-Finsler vector bundles

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Abstract. In the present paper, we shall be concerned with Einstein-Finsler bundles, and study the semi-stability of them.

§0. Introduction

Let $\pi : E \rightarrow M$ be a holomorphic vector bundle over a complex manifold M , and $p : PE \rightarrow M$ the projective bundle of E . If a complex Finsler structure F is given on E , a natural connection on the pull-back $\tilde{E} = p^{-1}E$ is defined, and the differential geometry of (E, F) has been studied (cf. ABATE–PATRIZIO [1], AIKOU [2, 3], FARAN[4], KOBAYASHI [5, 7], PATRIZIO–WONG [9], ROYDEN [10], RUND [11]).

In this paper, we are concerned with Einstein-Finsler bundles. This notion has been introduced by KOBAYASHI [7] as a natural generalization of Hermitian case, and the following problem is proposed:

What are algebro-geometric consequences of the Einstein-condition?

The original definition of Einstein-Finsler condition due to KOBAYASHI [7], however, has no invariant meanings as he pointed out himself. Hence the first purpose of this paper is to give the definition of Einstein-Finsler condition so that it has an invariant meaning, and to discuss Einstein-Finsler bundles (cf. Chapt. IV in KOBAYASHI [6]). We define Einstein-Finsler condition in terms of the curvature tensor of a partial

Mathematics Subject Classification: 53B40, 53C25.

Key words and phrases: complex Finsler structures, Einstein-Finsler structures, partial connections.

connection, not of a Hermitian connection. The second purpose is to discuss the semi-stability of Einstein-Finsler bundle over a compact Kähler manifold.

In §1, we recall the notion of complex Finsler structures on a holomorphic vector bundle and its *partial connection*, and in §2, we shall show some properties of its curvature which plays an important role in this paper.

In §3, we introduce the notion of *Einstein-Finsler bundles* in terms of the curvature tensor field of the partial connection, and in §4, we shall show a vanishing theorem of Bochner-type for holomorphic sections of complex Finsler bundles. In §5, we are concerned with the second fundamental form of partial connections. In §6, we shall discuss the semi-stability of Einstein-Finsler bundles over a compact Kähler manifold under a special condition.

We shall introduce some basic notations. Let M be a complex manifold of dimension n , and E a holomorphic vector bundle of rank r over M . Each fibre E_z is a complex vector space of complex dimension r . In the case of $r = 1$, any complex Finsler structure is Hermitian. Hence, throughout the remainder of the paper, we shall always assume $r > 1$. We denote by $p : PE \rightarrow M$ the projective bundle associated to E , and $\tilde{E}$ the induced bundle $p^{-1}E$ over PE :

$$\begin{array}{ccc} \tilde{E} & \xrightarrow{\tilde{p}} & E \\ \tilde{\pi} \downarrow & & \downarrow \pi \\ PE & \xrightarrow[p]{} & M \end{array}$$

We denote by LE the tautological line subbundle of $\tilde{E}$:

$$LE := \{(v, V) \in E \times PE; v \in V\}.$$

There exists a natural homomorphism $\tau : E \rightarrow LE$ which maps $E^\times$ biholomorphically to $LE^\times$, where $E^\times$ (resp. $LE^\times$) denotes the open submanifold of E (resp. LE) consisting of the non-zero elements.

Let $\{U, (z^i)\}$ be a complex coordinate system of M , and $\{\pi^{-1}(U), (z^\alpha, \xi^i)\}$ the induced complex coordinate system on E with respect to a holomorphic frame field $\{s_1, \dots, s_r\}$ on U .

Definition 0.1. A complex Finsler structure F on E is a real valued function satisfying the following conditions:

- (1) F is C^∞ -class on $E^\times$;
- (2) $F(z, \xi) \geq 0$, and “0” if and only if $\xi = 0$;
- (3) $F(z, \lambda\xi) = |\lambda|^2 F(z, \xi)$ for $\forall \lambda \in \mathbb{C}$.

If a complex Finsler structure F is given on E , the norm $\|\xi\|$ of an arbitrary section $\xi \in \Gamma(E)$ is defined by $\|\xi(z)\| = \sqrt{F(z, \xi(z))}$. Moreover, a Hermitian structure h on $L(E)$ is introduced by $h(\tau(\xi(z))) = F(z, \xi(z))$, and so there exists a one-to-one corresponding between Hermitian structures on LE and Finsler structures on E (cf. KOBAYASHI [5]).

We shall use the following notation throughout this paper:

- a^p (resp. A^p): the space of p -forms on M (resp. PE),
 $a^{p,q}$ (resp. $A^{p,q}$): the space of (p, q) -forms on M (resp. PE),
 $A^p(\tilde{E})$ (resp. $A^{p,q}(\tilde{E})$): the space of $\tilde{E}$ -valued p -forms (resp. (p, q) -forms) on PE ,
 $a^p(E)$ (resp. $a^{p,q}(E)$): the space of E -valued p -forms (resp. (p, q) -forms) on M .

§1. Finsler structures and partial connections

For any local holomorphic frame field $s = (s_1, \dots, s_r)$ of E , we denote by $(z, \xi) = (z^1, \dots, z^n, \xi^1, \dots, \xi^r)$ the induced local coordinate system on E , and by $(z, [\xi])$ the point of PE represented by (z, ξ) . A complex Finsler structure F is said to be *convex* if the Hermitian matrix $(F_{i\bar{j}})$ defined by

$$(1.1) \quad F_{i\bar{j}} := \frac{\partial^2 F}{\partial \xi^i \partial \bar{\xi}^j}$$

is positive-definite. In the following, we always assume the convexity of F . By the condition (3) in Definition 0.1, matrix components $F_{i\bar{j}}$ defined by (1.1) are functions on PE .

Putting $Z^i = \xi^i \circ p$, we take $(z, [\xi], Z) = (z^1, \dots, z^n, \xi^1 : \dots : \xi^r, Z^1, \dots, Z^r)$ as a local coordinate system for $\tilde{E}$. Then, for $\forall Z, W \in A^0(\tilde{E})$, a Hermitian structure H on $\tilde{E}$ is defined by

$$H(Z, W) := \sum_{i,j} F_{i\bar{j}} Z^i \bar{W}^j.$$

The Hermitian connection $\nabla : A^0(\tilde{E}) \rightarrow A^1(\tilde{E})$ in $(\tilde{E}, H)$ is given by the form

$$(1.2) \quad \theta_j^i := \sum_l F^{\bar{l}i} \partial F_{j\bar{l}} = \sum_{l,\alpha} F^{\bar{l}i} \frac{\partial F_{j\bar{l}}}{\partial z^\alpha} dz^\alpha + \sum_{l,k} F^{\bar{l}i} \frac{\partial F_{j\bar{l}}}{\partial \xi^k} d\xi^k.$$

A Finsler structure F on E is a Hermitian structure on E if and only if the second term in (1.2) vanishes identically.

We shall introduce a *partial connection* in $(\tilde{E}, H)$ which is the main tool in this paper. The partial connection D is defined as a covariant derivation in transversal direction to the fibres of PE . For the cotangent bundle T_{PE}^* of PE , we shall introduce a C^∞ -splitting by defining the left splitting σ of the exact sequence

$$0 \longrightarrow \mathcal{H}^* \longrightarrow T_{PE}^* \underset{\sigma}{\overset{\longleftarrow}{\longrightarrow}} (\ker dp)^* \longrightarrow 0$$

by

$$\sigma(d\xi^i) := d\xi^i + \sum_j \theta_j^i \xi^j.$$

The set of local 1-forms $\{dz^1, \dots, dz^n, \theta^1, \dots, \theta^r\}$, $\theta^i := \sigma(d\xi^i)$, is a local co-frame field for T_{PE}^* , and it defines a C^∞ -splitting

$$(1.3) \quad T_{PE}^* = \mathcal{H}^* \oplus \mathcal{V}^*,$$

where $\mathcal{H}^*$ is locally spanned by $\{dz^1, \dots, dz^n\}$, and $\mathcal{V}^*$ by $\{\theta^1, \dots, \theta^r\}$. We shall denote by $p'_\mathcal{H} : A^0(T_{PE}^*) \rightarrow A^0(\mathcal{H}^*)$ and $p''_\mathcal{H} : A^0(T_{PE}^*) \rightarrow A^0(\mathcal{V}^*)$ the natural projections with respect to the splitting (1.3) respectively. Then we shall define a homomorphism $D := D' + D'' : A^0(\tilde{E}) \rightarrow A^0((\mathcal{H} \oplus \mathcal{H})^* \otimes \tilde{E})$ so that the following diagram is commutative.

$$\begin{array}{ccc} A^0(\tilde{E}) & \xrightarrow{\nabla} & A^0((T_{PE} \oplus \bar{T}_{PE})^* \otimes \tilde{E}) \\ \parallel & & \downarrow p_\mathcal{H} \otimes 1 \\ A^0(\tilde{E}) & \xrightarrow{D} & A^0((\mathcal{H} \oplus \bar{\mathcal{H}})^* \otimes \tilde{E}) \end{array}$$

For $\forall Z \in A^0(\tilde{E})$, we shall define

$$DZ := (p_\mathcal{H} \otimes 1) \circ \nabla Z.$$

In the following, we shall show the local expression of D . For this purpose, we rewrite the form θ_j^i as follows:

$$(1.4) \quad \theta_j^i = \sum_{\alpha} \Gamma_{j\alpha}^i dz^{\alpha} + \sum_k C_{jk}^i \theta^k,$$

where we put

$$\begin{aligned} \Gamma_{j\alpha}^i &:= \sum_l F^{\bar{l}i} \left(\frac{\partial F_{j\bar{l}}}{\partial z^{\alpha}} - \sum_m N_{\alpha}^m \frac{\partial F_{j\bar{l}}}{\partial \xi^m} \right), \quad C_{jk}^i := \sum_l F^{\bar{l}i} \frac{\partial F_{j\bar{l}}}{\partial \xi^k}, \\ N_{\alpha}^m &:= \sum_j \xi^j \Gamma_{j\alpha}^m = \sum_{m,r} F^{\bar{r}m} \frac{\partial F_{l\bar{r}}}{\partial z^{\alpha}} \xi^l. \end{aligned}$$

We define operators $\partial_{\mathcal{H}}, \partial_{\mathcal{V}} : A^0 \rightarrow A^{1,0}$ by $\partial_{\mathcal{H}} := p'_{\mathcal{H}} \circ \partial$, and by $\partial_{\mathcal{V}} := \partial - \partial_{\mathcal{H}}$ respectively. Then

$$\begin{aligned} \partial_{\mathcal{H}} f &:= \sum_{\alpha} \frac{\delta f}{\delta z^{\alpha}} dz^{\alpha} := \sum_{\alpha} \left(\frac{\partial f}{\partial z^{\alpha}} - \sum_m N_{\alpha}^m \frac{\partial f}{\partial \xi^m} \right) dz^{\alpha}, \\ \partial_{\mathcal{V}} f &:= \sum_m \frac{\partial f}{\partial \xi^m} \theta^m = \partial f - \partial_{\mathcal{H}} f \end{aligned}$$

for $\forall f \in A^0$ respectively. Then, by definition of $\partial_{\mathcal{H}}$, we get the following:

Proposition 1.1. *For the given complex Finsler structure, the following identity holds:*

$$\partial_{\mathcal{H}} F \equiv 0.$$

In terms of $\partial_{\mathcal{H}}$, the first term of (1.4) is written as follows:

$$(1.5) \quad \omega_j^i := \sum_{\alpha} \Gamma_{j\alpha}^i dz^{\alpha} = \sum_r F^{\bar{r}i} \partial_{\mathcal{H}} F_{j\bar{r}}.$$

The $(1,0)$ -part $D' : A^0(\tilde{E}) \rightarrow A^{1,0}(\tilde{E})$ and the $(0,1)$ -part $D'' : A^0(\tilde{E}) \rightarrow A^{0,1}(\tilde{E})$ of D are given by

$$D'Z := (p'_{\mathcal{H}} \otimes 1)(\nabla Z), \quad D''Z := (p''_{\mathcal{H}} \otimes 1)(\nabla Z)$$

respectively. Their local expression are given by

$$D'Z = \sum_i \left(\partial_{\mathcal{H}} Z^i + \sum_m Z^m \omega_m^i \right) \otimes s_i := \sum_{\alpha} (D_{\alpha} Z^i) dz^{\alpha} \otimes s_i,$$

$$D''Z = \sum_i \bar{\partial}_{\mathcal{H}} Z^i \otimes s_i$$

for $\forall Z = \sum_i Z^i s_i \in A^0(\tilde{E})$. $D = D' + D''$ is a *partial connection* in the following sense.

Proposition 1.2. *Let (E, F) be a complex Finsler bundle. Then the homomorphism $D : A^0(\tilde{E}) \rightarrow A^1(\tilde{E})$ is uniquely determined in (E, F) and satisfies*

$$D(fZ) = d_{\mathcal{H}} f \otimes Z + f DZ$$

for $\forall f \in A^0$ and $\forall Z \in A^0(\tilde{E})$, where we put $d_{\mathcal{H}} := \partial_{\mathcal{H}} + \bar{\partial}_{\mathcal{H}}$.

PROOF. Since the operator $\partial_{\mathcal{H}}$ is uniquely determined from F , and by (1.5), the operator D is also determined uniquely from F . The second assertion is obvious from its definition. $\square$

The Hermitian connection ∇ in $(\tilde{E}, H)$ is metrical:

$$dH(Z, W) = H(\nabla Z, W) + H(Z, \nabla W).$$

The partial connection D , however, is partially metrical, that is,

Proposition 1.3. *For $\forall Z, W \in A^0(\tilde{E})$, the following identities hold:*

$$(1.6) \quad d_{\mathcal{H}} H(Z, W) = H(DZ, W) + H(Z, DW),$$

$$(1.7) \quad D' F_{i\bar{j}} := \partial_{\mathcal{H}} F_{i\bar{j}} - \sum_m F_{m\bar{j}} \omega_i^m = 0.$$

§2. Curvature of partial connection D

In this section, we shall state some important properties of the partial connection D . For this purpose, we shall prepare some lemmas.

First, extending the operator $\partial_{\mathcal{H}}$ to the space A^p , we have

Lemma 2.1. *The $(1,0)$ -form ω_j^i defined by (1.5) satisfies*

$$\partial_{\mathcal{H}}\omega_j^i + \sum_r \omega_r^i \wedge \omega_j^r = 0.$$

PROOF. This equality is obtained by direct calculations. In fact, if we write

$$\partial_{\mathcal{H}}\omega_j^i + \sum_r \omega_r^i \wedge \omega_j^r := \sum_{\alpha,\beta} R_{j\alpha\beta}^i dz^\alpha \wedge dz^\beta,$$

we get

$$R_{j\alpha\beta}^i = \frac{\delta}{\delta z^\alpha} \Gamma_{j\beta}^i - \frac{\delta}{\delta z^\beta} \Gamma_{j\alpha}^i + \sum_l \Gamma_{l\alpha}^i \Gamma_{j\beta}^l - \sum_l \Gamma_{l\beta}^i \Gamma_{j\alpha}^l.$$

From $\Gamma_{j\alpha}^i = \sum_l F^{\bar{m}i} \delta F_{j\bar{m}} / \delta z^\alpha$, direct calculations give

$$R_{j\alpha\beta}^i = \sum_{l,m} C_{jl}^i R_{\alpha\beta}^l,$$

where we put

$$R_{\alpha\beta}^i := \frac{\delta N_\alpha^i}{\delta z^\beta} - \frac{\delta N_\beta^i}{\delta z^\alpha}.$$

On the other hand, by definition, we get also the following relation:

$$\sum_j \xi^j R_{j\alpha\beta}^i = R_{\alpha\beta}^i.$$

These relations and the identity $\sum_j \xi^j C_{jl}^i \equiv 0$ imply $R_{j\alpha\beta}^i \equiv 0$. □

By the proof above, we also have the following

Lemma 2.2. $\partial_{\mathcal{H}}^2 f \equiv 0$ for $\forall f \in A^0$.

PROOF. By direct calculations, we have

$$\begin{aligned} \partial_{\mathcal{H}}^2 f &= \frac{1}{2} \sum_{m,\alpha,\beta} \left(\frac{\delta N_\beta^m}{\delta z^\alpha} - \frac{\delta N_\alpha^m}{\delta z^\beta} \right) \frac{\partial f}{\partial \xi^m} dz^\alpha \wedge dz^\beta \\ &= \frac{1}{2} \sum_{m,\alpha,\beta} R_{\alpha\beta}^m \frac{\partial f}{\partial \xi^m} dz^\alpha \wedge dz^\beta. \end{aligned}$$

Since $R_{\alpha\beta}^m \equiv 0$, we get $\partial_{\mathcal{H}}^2 f \equiv 0$. □

Secondly we shall extend D to the space $A^p(\tilde{E})$ in the usual way, that is, for $\forall \sum_i \phi(z, [\xi])^i \otimes s_i \in A^p(\tilde{E})$,

$$\begin{aligned} D' \left(\sum_i \phi^i \otimes s_i \right) &= \sum_i \partial_{\mathcal{H}} \phi^i \otimes s_i + (-1)^p \sum_i \phi^i \wedge D' s_i, \\ D'' \left(\sum_i \phi^i \otimes s_i \right) &= \sum_i \bar{\partial}_{\mathcal{H}} \phi^i \otimes s_i. \end{aligned}$$

Then we introduce an $\text{End}(\tilde{E})$ -valued $(1, 1)$ -form $R = (R_j^i)$ by

$$(2.1) \quad \Omega_j^i := \bar{\partial}_{\mathcal{H}} \omega_j^i = \sum_{\alpha, \beta} R_{j\alpha\bar{\beta}}^i dz^\alpha \wedge d\bar{z}^\beta,$$

where we put

$$R_{j\alpha\bar{\beta}}^i := -\frac{\delta \Gamma_{j\alpha}^i}{\delta \bar{z}^\beta} = -\sum_m F^{\bar{m}i} \left(\frac{\delta^2 F_{j\bar{m}}}{\delta z^\alpha \delta \bar{z}^\beta} - \sum_{k,l} F^{\bar{l}k} \frac{\delta F_{j\bar{l}}}{\delta z^\alpha} \frac{\delta F_{k\bar{m}}}{\delta \bar{z}^\beta} \right).$$

Then we have the following fundamental theorem.

Theorem 2.1. *The partial connection $D = D' + D''$ satisfies*

$$D' D' \equiv 0, \quad D'' D'' \equiv 0, \quad DD = D' D'' + D'' D',$$

and for $\forall Z \in A^0(\tilde{E})$

$$(2.2) \quad DDZ = R(Z),$$

where $R \in A^{1,1}(\text{End}(\tilde{E}))$ is defined by

$$R(Z) = \sum_{i,j} Z^j \Omega_j^i \otimes s_i.$$

PROOF. The first assertion is directly derived from Lemma 2.1 and 2.2. We shall prove the equation (2.2) only. Since $DD(fZ) = fDDZ$ for $\forall f \in A^0$, it is sufficient to prove $DDs_i = \sum_m \Omega_i^m s_m$. By definition of D' and D'' , we get

$$\begin{aligned} (D' D'' + D'' D') s_i &= D'' D' s_i = D'' \left(\sum_m \omega_i^m s_m \right) \\ &= \bar{\partial}_{\mathcal{H}} \left(\sum_m \omega_i^m s_m \right) = \sum_m \Omega_i^m s_m. \end{aligned}$$

So we have completed the proof. $\square$

Proposition 2.1. *The partial connection D satisfies*

$$\partial_{\mathcal{H}}\bar{\partial}_{\mathcal{H}}H(Z, Z) = H(DZ, DZ) - H(R(Z), Z)$$

for any holomorphic section Z of $\tilde{E}$,

In this paper, we call R the *curvature* of D .

§3. Einstein-Finsler condition

Let (E, F) be a complex Finsler bundle over a Hermitian manifold (M, g) , $g = \sum_{\alpha, \beta} g_{\alpha\bar{\beta}}(z) dz^{\alpha} \otimes d\bar{z}^{\beta}$. We shall define the *mean curvature* $K_{i\bar{j}}$ of (E, F) by the partial mean curvature of $(\tilde{E}, H)$, that is, the g -trace

$$K_j^i := \sum_{\alpha, \bar{\beta}} g^{\bar{\beta}\alpha} R_{j\alpha\bar{\beta}}^i,$$

where $R_{j\alpha\bar{\beta}}^i$ is the *curvature tensor* defined by (2.1). Putting $K_{i\bar{j}} := \sum_m F_{m\bar{j}} K_i^m$, we shall define a Hermitian form K by

$$K(Z, W) = \sum_{i, j} K_{i\bar{j}} Z^i \bar{W}^j$$

for $\forall Z, W \in A^0(\tilde{E})$.

Definition 3.1. A complex Finsler bundle (E, F) is said to be *weakly Einstein-Finsler* if the partial mean curvature K of $(\tilde{E}, H)$ satisfies

$$(3.1) \quad K_{i\bar{j}} = \varphi(z) F_{i\bar{j}}$$

for a function φ on M . If the factor φ is constant, (E, F) is said to be *Einstein-Finsler*.

Remark 3.1. (1) We note that the original definition of K_j^i of KOBAYASHI [7] has no invariant meaning, since the quantities $R_{j\alpha\bar{\beta}}^i$ in [7] is not a tensor field (See (5.6) in [7]). Our definition of K_j^i , however, has an invariant meaning because of the tensority of $R_{j\alpha\bar{\beta}}^i$.

(2) If the given F is a Hermitian structure, that is, $F(z, \xi) = \sum_{i,j} h_{i\bar{j}}(z) \xi^i \bar{\xi}^j$, the curvature $\Omega_j^i = \sum_{\alpha,\beta} R_{j\alpha\bar{\beta}}^i dz^\alpha \wedge d\bar{z}^\beta$ of D is just the one of $h = (h_{i\bar{j}})$. Hence our definition is a natural generalization of Hermitian case.

In the rest of this section, we are concerned with conformal changes of complex Finsler structures. A conformal change of F is defined by $F \rightarrow \tilde{F} := e^{u(z)} F$ for a smooth function $u(z)$ on M .

Lemma 3.1. *The curvature $\tilde{R}$ of $\tilde{D}$ in $(E, \tilde{F})$ is given by*

$$\tilde{\Omega}_j^i = \Omega_j^i + \bar{\partial} \partial u \delta_j^i.$$

PROOF. Since the connection form $\tilde{\omega}_j^i$ of $\tilde{D}$ is given by $\tilde{\omega}_j^i = \omega_j^i + \partial u \delta_j^i$, the functions $\tilde{N}_\alpha^i$ derived from $\tilde{F}$ is given by

$$\tilde{N}_\alpha^i = N_\alpha^i + \frac{\partial u}{\partial z^\alpha} \xi^i.$$

Hence the operator $\partial_{\tilde{\mathcal{H}}}$ satisfies

$$\partial_{\tilde{\mathcal{H}}} f = \partial_{\mathcal{H}} f - \sum_m \xi^m \frac{\partial f}{\partial \xi^m} \partial u$$

for $\forall f \in A^0$. By this relation and the formula

$$\sum_m \frac{\partial \Gamma_{j\alpha}^i}{\partial \xi^m} \bar{\xi}^m \equiv 0,$$

we get easily our assertion. $\square$

By this lemma, the mean curvature $\tilde{K}_j^i$ of $(E, \tilde{F})$ is given by

$$(3.2) \quad \tilde{K}_j^i = K_j^i + \square u \delta_j^i,$$

where we put

$$\square u := - \sum_{\alpha,\beta} g^{\bar{\beta}\alpha} \frac{\partial^2 u}{\partial z^\alpha \partial \bar{z}^\beta}.$$

Hence we have

Proposition 3.1. *Let (E, F) be a weakly Einstein-Finsler bundle with factor φ . Then $(E, e^{u(z)}F)$ satisfies the Einstein-Finsler condition (3.1) with factor $\varphi + \square u$.*

Moreover we have

Proposition 3.2. *Let (E, F) be a weakly Einstein-Finsler bundle over a compact Kähler manifold (M, Φ) . Then there exists a conformal change of $F \rightarrow \tilde{F}$ such that $(E, \tilde{F})$ is an Einstein-Finsler bundle.*

PROOF. The proof is similar to Proposition 2.4 of Kobayashi [6]. For the constant c defined by

$$c \int_M \Phi^n = \int_M \varphi \Phi^n,$$

the harmonic theory on M implies that there exists a function $u(z)$ on M satisfying $c - \varphi(z) = \square u$. If we consider the conformal change $F \rightarrow \tilde{F} := e^{u(z)}F$ for this function $u(z)$, Proposition 3.1 implies $\tilde{K}_j^i = c\delta_j^i$, and so $(E, \tilde{F})$ is an Einstein-Finsler bundle with constant factor c . $\square$

Remark 3.2. In the case of $F = \sum_{i,j} h_{i\bar{j}}(z) \xi^i \bar{\xi}^j$, the constant c is given by

$$(3.3) \quad c = \frac{\int_M \varphi \Phi^n}{\int_M \Phi^n} = \frac{2n\pi \deg(E)}{r \cdot \text{vol}(M)},$$

where the degree $\deg(E)$ of E is defined by

$$\deg(E) := \int_M c_1(E) \wedge \Phi^{n-1}.$$

Hence the constant c depends only on the cohomology class of Φ and the first Chern class $c_1(E)$, not on the Hermitian structure h (cf. KOBAYASHI [6], p. 104).

The mean curvature K_j^i in (3.1) may be considered as an endomorphism of the bundle E . We do not know whether there exists an Hermitian structure on E whose mean curvature is K_j^i in (3.1), except the case where (E, F) is modeled on a complex Minkowski space (cf. §6). Hence, in general, the constant c in Proposition 3.4 depends on the given Finsler structure F .

In general, for the given two Finsler bundles (E, F) and (E', F') , we do not know the natural way to define a Finsler structure on the tensor

product $E \otimes E'$ in a computable form. In the case where E' is a line bundle L , however, we can define a Finsler structure $F^{E \otimes L}$ on $E \otimes L$ as follows. Since any Finsler structure F^L on a line bundle L is a Hermitian structure, for any section λ of L we may put

$$(3.4) \quad F^L(z, \lambda) = a(z) |\lambda|^2,$$

where $a(z)$ is a positive-valued C^∞ -function. Then, for $\forall \xi = \sum \xi^i s_i \otimes t \in a^0(E \otimes L)$, we shall define $F^{E \otimes L}$ by

$$F^{E \otimes L}(z, \xi) := a(z) F(z, \xi).$$

Then, by Lemma 3.1, the curvature $R^{E \otimes L}$ of $F^{E \otimes L}$ is given by

$$R^{E \otimes L} = R^E \otimes 1 + I_E \otimes \bar{\partial} \partial \log a(z),$$

and the mean curvature $K^{E \otimes L}$ is given by

$$K^{E \otimes L} = K^E \otimes 1 + I_E \otimes \square \log a(z).$$

Hence we have

Proposition 3.3. *Let (E, F) be a weak Einstein-Finsler bundle with factor φ , and (L, F^L) an arbitrary line bundle with a Hermitian metric (3.4). Then the tensor product $E \otimes L$ admits a weak Einstein-Finsler structure with factor $\varphi + \square \log a(z)$.*

§4. A vanishing theorem for holomorphic sections

In this section, we shall show a Bochner-type vanishing theorem for holomorphic sections of complex Finsler bundle (E, F) .

Let $\zeta = \sum_i \zeta^i(z) s_i$ be a non-vanishing holomorphic section over an open set U . We denote by $PE_{\zeta(U)} \subset PE$ the image of $\zeta(U)$ by the natural projection $E^\times \rightarrow PE$, that is,

$$PE_{\zeta(U)} := \{(z, [\zeta(z)]) \in PE; z \in U\}.$$

We also denote by ζ_P the corresponding holomorphic section of LE over $PE_{\zeta(U)}$. For the holomorphic mapping $f_\zeta : z \in U \rightarrow (z, [\zeta(z)]) \in PE$, we

get the following commutative diagram:

$$\begin{array}{ccc} LE^\times & \xleftarrow{\tau} & E^\times \\ \zeta_P \uparrow & & \uparrow \zeta \\ PE_{\zeta(U)} & \xleftarrow{f_\zeta} & U \end{array}$$

We say that a holomorphic section $\zeta = \sum_i \zeta^i(z) s_i$ is *parallel with respect to D* if it satisfies $D\zeta_P = 0$ on $PE_{\zeta(U)}$, that is,

$$(4.1) \quad D_\alpha \zeta^i := \frac{\partial \zeta^i}{\partial z^\alpha} + \sum_{m=1}^r \zeta^m(z) \Gamma_{m\alpha}^i(z, [\zeta(z)]) = 0.$$

For any holomorphic section ζ of E , we show the following Weitzenböck-type formula.

Proposition 4.1. *Let (E, F) be a complex Finsler bundle over a Hermitian manifold (M, g) . For any holomorphic section ζ of E , the following identity holds:*

$$\square F(z, \zeta(z)) = \|D'\zeta_P\|^2 - K(\zeta_P, \zeta_P),$$

where

$$\|D'\zeta_P\|^2 := \sum_{\alpha, \beta, i, j} g^{\bar{\beta}\alpha}(z) F_{i\bar{j}}(z, [\zeta(z)]) D_\alpha \zeta^i \overline{D_\beta \zeta^j}.$$

PROOF. For any function f , we have $D''D'f = \bar{\partial}_{\mathcal{H}}\partial_{\mathcal{H}}f$. We shall apply this to the function $f(z) = F(z, \zeta(z)) = H(\zeta_P, \zeta_P)$. Proposition 2.1 implies

$$\partial_{\mathcal{H}}\bar{\partial}_{\mathcal{H}}H(\zeta_P, \zeta_P) = -H(R(\zeta_P), \zeta_P) + H(D'\zeta_P, D'\zeta_P).$$

Hence we get

$$\partial\bar{\partial}f = -H(R(\zeta_P), \zeta_P) + H(D'\zeta_P, D'\zeta_P).$$

By taking the g -trace of the equation above, we complete the proof. $\square$

By this formula and the maximum principle of E. Hopf (cf. Theorem 1.10 in p. 52 of KOBAYASHI [6]), we can show the following Bochner-type vanishing theorem for holomorphic sections:

Theorem 4.1. *Let (E, F) be a complex Finsler bundle over a compact Hermitian manifold (M, g) .*

- (1) *If the mean curvature K is negative semi-definite on PE , then every holomorphic section ζ of E is parallel with respect to D , that is,*

$$D\zeta_P = 0,$$

and satisfies

$$K(\zeta_P, \zeta_P) = 0.$$

- (2) *If K is negative semi-definite on PE and negative definite at some point of PE , then E admits no nonzero holomorphic sections.*

By this theorem, we have

Proposition 4.2. *Let (E, F) be an Einstein-Finsler bundle over a compact Hermitian manifold with constant factor φ .*

- (1) *If $\varphi = 0$, then every non-vanishing holomorphic section of E is parallel with respect to D .*
 (2) *If $\varphi < 0$, then E admits no nonzero holomorphic sections.*

§5. Partial second fundamental form

In this section, we shall define a $(1, 0)$ -form which plays a role of the so-called second fundamental form. Let (E, F) be a Finsler vector bundle over a compact Kähler manifold (M, Φ) with a convex Finsler structure F . The Hermitian structure and the partial connection on the induced bundle $\tilde{E}$ is also denoted by H and by D respectively.

Let S be a holomorphic subbundle of E with rank s , and $\tilde{S}$ the induced bundle $p^{-1}S$ over PE . We denote by H_S the restriction of H to $\tilde{S}$, and by D_S the partial connection on $(\tilde{S}, H_S)$. Then we define

$$A(Z) := (D - D_S)Z$$

for a section $Z \in A^0(\tilde{S})$. For the quotient bundle $\tilde{Q} := \tilde{E}/\tilde{S}$, it is proved easily that A is an $\text{End}(\tilde{S}, \tilde{Q})$ -valued $(1, 0)$ -form. We shall call A the *partial second fundamental form* of (E, F_S) . We say that a section Z of $\tilde{E}$ is *partial-holomorphic* if it satisfies

$$D''Z = \bar{\partial}_{\mathcal{H}}Z = 0.$$

Then we have

Proposition 5.1. *The partial second fundamental form A vanishes identically if and only if the exact sequence*

$$(5.1) \quad 0 \rightarrow \tilde{S} \rightarrow \tilde{E} \rightarrow \tilde{Q} \rightarrow 0$$

splits H -orthogonally and partial-holomorphically.

PROOF. Suppose $A \equiv 0$. This assumption implies $D|_{\tilde{S}} = D_S$, and so $D(A^0(\tilde{S})) \subset A^1(\tilde{S})$. We decompose $\tilde{E}$ as $\tilde{E} = \tilde{S} \oplus \tilde{S}^\perp$, where $\tilde{S}^\perp \cong \tilde{Q}$ is the H -orthogonal complement. For $\forall \xi \in A^0(\tilde{S})$, $\xi^\perp \in A^0(\tilde{S}^\perp)$, we have

$$H(D\xi, \xi^\perp) + H(\xi, D\xi^\perp) = d_{\mathcal{H}}H(\xi, \xi^\perp) = 0.$$

Thus we get $D(A^0(\tilde{S}^\perp)) \subset A^1(\tilde{S}^\perp)$. For an arbitrary holomorphic section σ of $\tilde{E}$, we write

$$\sigma = \xi + \xi^\perp$$

where $\xi \in A^0(\tilde{S})$, $\xi^\perp \in A^0(\tilde{S}^\perp)$. Since σ is holomorphic, $\bar{\partial}_{\mathcal{H}}\sigma = \bar{\partial}_{\mathcal{H}}\xi + \bar{\partial}_{\mathcal{H}}\xi^\perp = 0$. Moreover $\bar{\partial}_{\mathcal{H}} = D''$ implies $\bar{\partial}_{\mathcal{H}}\xi \in A^{0,1}(\tilde{S})$, $\bar{\partial}_{\mathcal{H}}\xi^\perp \in A^{0,1}(\tilde{S}^\perp)$. Consequently we get

$$\bar{\partial}_{\mathcal{H}}\xi = 0, \quad \bar{\partial}_{\mathcal{H}}\xi^\perp = 0,$$

which means that the splitting $\tilde{E} = \tilde{S} \oplus \tilde{S}^\perp$ is partial-holomorphically.

Conversely, if we denote by H_Q the restriction of H to the holomorphic bundle $\tilde{Q}$, the partial connections $D_S \oplus D_Q$ of $(\tilde{S} \oplus \tilde{Q}, H_S \oplus H_Q)$ defines the one of $(\tilde{E}, H)$. Thus we get $A \equiv 0$. $\square$

By direct calculations, the curvature form Ω of D can be written in the following form:

$$(5.2) \quad \Omega := \begin{pmatrix} \Omega_S + A \wedge {}^t\bar{A} & * \\ * & \Omega_Q + {}^t\bar{A} \wedge A \end{pmatrix},$$

where Ω_S and Ω_Q is the curvature form of the partial connection induced on the bundle $\tilde{S}$ and $\tilde{Q}$ respectively. Then we have

$$H(\Omega(X, \bar{X})Z, Z) = H_S(\Omega_S(X, \bar{X})Z, Z) - \|A(X)Z\|^2$$

for $\forall X \in TPE$ and $\forall Z \in A^0(\tilde{S})$.

Now we shall introduce an analogy of first Chern form of vector bundle. We shall set as

$$c(\tilde{E}) := \frac{\sqrt{-1}}{2\pi} \sum_{j=1}^r \Omega_j^j, \quad c(\tilde{S}) := \frac{\sqrt{-1}}{2\pi} \sum_{j=1}^s \Omega_S^j.$$

Moreover, for a non-vanishing holomorphic section ζ of E on an open set $U \subset M$, we shall consider the pull-back of $c(\tilde{E})$ and $c(\tilde{S})$ by f_ζ :

$$c_\zeta(E) := f_\zeta^* c(\tilde{E}), \quad c_\zeta(S) := f_\zeta^* c(\tilde{S}).$$

$c_\zeta(E)$ and $c_\zeta(S)$ are $(1,1)$ -forms on U . For the Kähler form $\Phi = \sqrt{-1} \sum g_{\alpha\bar{\beta}}(z) dz^\alpha \wedge d\bar{z}^\beta$ on M , we know the following (cf. KOBAYASHI [6], p. 55):

$$\begin{aligned} d_\zeta(E) &:= c_\zeta(E) \wedge \Phi^{n-1} = \frac{1}{2n\pi} \tilde{g} \left(f_\zeta^* \sum_{j=1}^r \Omega_j^j \right) \Phi^n, \\ d_\zeta(S) &:= c_\zeta(S) \wedge \Phi^{n-1} = \frac{1}{2n\pi} \tilde{g} \left(f_\zeta^* \sum_{j=1}^s \Omega_S^j \right) \Phi^n, \end{aligned}$$

where the notion $\tilde{g}$ means that $\tilde{g}(\sigma) := \sum g^{\alpha\bar{\beta}} \sigma_{\alpha\bar{\beta}}$ for an arbitrary $(1,1)$ -form $\sigma = \sum \sigma_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta$ on M .

Proposition 5.2. *Let (E, F) be an Einstein-Finsler bundle with constant factor φ over a compact Kähler manifold (M, Φ) . For any holomorphic subbundle S of E , we have*

$$(5.3) \quad \frac{d_\zeta(S)}{\text{rank } S} \leq \frac{d_\zeta(E)}{\text{rank } E}$$

for any non-vanishing holomorphic section ζ . If the equality holds, the exact sequence (5.1) splits partial-holomorphically, and S, Q are Einstein-Finsler vector bundle with the same factor φ .

PROOF. From (5.2) we get

$$\frac{2\pi}{\sqrt{-1}} c(\tilde{S}) = \sum_{j=1}^s \Omega_j^j - \sum_{\lambda=1}^s \sum_{\mu=1}^{r-s} A_\lambda^\mu \wedge \bar{A}_\lambda^\mu,$$

where we put $A = (A_\lambda^\mu)$, $A_\lambda^\mu = \sum_{\alpha=1}^n A_{\lambda\alpha}^\mu dz^\alpha$. The Einstein condition (3.1) implies $\tilde{g}(f_\zeta^* \sum_{j=1}^r \Omega_j^j) = r\varphi$. Hence we get

$$\frac{d_\zeta(E)}{r} = \frac{\varphi}{2n\pi} \Phi^n,$$

$$\frac{d_\zeta(S)}{s} = \frac{\varphi}{2n\pi} \Phi^n - \frac{1}{s} \tilde{g} \left(f_\zeta^* \sum_{\lambda=1}^s \sum_{\mu=1}^{r-s} A_\lambda^\mu \wedge \bar{A}_\lambda^\mu \right) \Phi^n.$$

Because of $\tilde{g}(f_\zeta^* A \wedge {}^t \bar{A}) \geq 0$, we get (5.3). The equality holds if and only if $A \equiv 0$. The second assertion is obtained from Proposition 5.1. $\square$

If ζ is defined on M , the constant factor φ is given by

$$(5.4) \quad \varphi = \frac{2n\pi}{r \cdot \text{vol}(M)} \int_M d_\zeta(E).$$

§6. Semi-stability: Special Cases

In this section, we shall consider the semi-stability of Einstein-Finsler bundles. We recall the definition of semi-stability in the sense of Mumford-Takemoto (cf. KOBAYASHI [6], p. 134).

Let $\mathcal{O}(E) := \mathcal{E}$ be the sheaf of germs of holomorphic sections of E . E is said to be Φ -stable (resp. Φ -semi-stable) if for any coherent subsheaf $\mathcal{F} \subset \mathcal{E}$ with $0 < \text{rank } \mathcal{F} < \text{rank } E$, the following inequality holds:

$$\mu(\mathcal{F}) := \frac{\deg(\mathcal{F})}{\text{rank } \mathcal{F}} < \frac{\deg(E)}{\text{rank } E} = \mu(E)$$

(resp. $\leq$). LÜBKE [8] gave a proof of the following

Theorem 6.1 (Kobayashi's theorem). *Let (E, h) be an Einstein-Hermitian vector bundle over a compact Kähler manifold (M, Φ) . Then E is Φ -semi-stable and (E, h) is a direct sum*

$$(E, h) = (E_1, h_1) \oplus \cdots \oplus (E_k, h_k)$$

of Φ -stable Einstein-Hermitian vector bundles $(E_1, h_1), \dots, (E_k, h_k)$ with the same factor c as (E, h) .

As shown in Proposition 3.3, for any (weak) Einstein-Finsler bundle (E, F) and Hermitian line bundle (L, h) , the tensor product $E \otimes L$ admits a

natural (weak) Einstein-Finsler structure. Then, by the idea of LÜBKE [8], we get the following

Proposition 6.1. *Let (E, F) be an Einstein-Finsler bundle over a compact Kähler manifold (M, Φ) with constant factor φ . Let $\mathcal{F}$ be any reflexive subsheaf of $\mathcal{E}$ of rank $\mathcal{F} = 1$, i.e. line bundle $\mathcal{F}$. Then the following inequality holds:*

$$\varphi \geq \frac{2n\pi}{\text{vol}(M)}\mu(\mathcal{F}).$$

PROOF. Since $\mathcal{F}$ is of rank $\mathcal{F} = 1$, it may be considered as (the sheaf of germs of holomorphic sections of) a holomorphic line bundle L . For any holomorphic line bundle over a compact Kähler manifold (M, Φ) , we may consider it as Einstein-Hermitian with constant factor $\psi = \frac{2n\pi}{\text{vol}(M)}\mu(L)$. The monomorphism $\mathcal{F} \rightarrow \mathcal{E}$ induce a non-trivial holomorphic section $f : \mathcal{O}_M \rightarrow \mathcal{E} \otimes \mathcal{F}^*$, which is considered as a global non-trivial holomorphic section of $E \otimes L^*$. By Proposition 3.3, $E \otimes L^*$ is an Einstein-Finsler bundle with constant factor $\varphi - \psi$. Since f is a non-trivial holomorphic section of $E \otimes L^*$, Proposition 4.2 completes the proof.

If the equality hold, since the bundle $E \otimes L^*$ is Einstein-Finsler with its factor $\varphi - \psi = 0$, Proposition 4.2 implies that the holomorphic section f is parallel with respect to the partial connection of $E \otimes L^*$, that is, L is parallel with respect to D . Then the pull pack $\tilde{E}$ splits partial-holomorphically as $\tilde{E} = \tilde{E}' \oplus \tilde{L}$. $\square$

At this time, we have no information about the semi-stability of Einstein-Finsler bundles. The difficulty lies in the facts that, for the case of rank $\mathcal{F} > 1$, there exists no computable way to define a Finsler structure on the tensor product $\otimes E$ from the given Finsler structure F on E , and that the constant φ depends on the given Finsler structure F .

On the other hand, if it is always possible to find a Hermitian structure on E such that its mean curvature is given by K_j^i in (3.1), then any Einstein-Finsler bundle is Φ -semi-stable. In general, we do not know the existence of such a Hermitian structure. So we shall treat special cases which are reducible to the case of Hermitian-Einstein.

§6.1. **Special case I.** We recall the following definition (cf. AIKOU [2, 3]).

Definition 6.1. A complex Finsler bundle (E, F) is said to be *modeled on a complex Minkowski space* if its partial connection D is induced from a connection in E , that is, $\Gamma_{j\alpha}^i = \Gamma_{j\alpha}^i(z)$.

In a previous papers [2], we have proved

Theorem 6.2. *Let (E, F) be a complex Finsler bundle which is modeled on a complex Minkowski space. Then there exists a Hermitian structure h_F in E such that D is induced from the Hermitian connection of h_F .*

Example 6.1. Let (E, h) be a *reducible* Hermitian vector bundle in the following sense, that is, we suppose that (E, h) splits holomorphically as an h -orthogonal sum

$$(E, h) = (E', h') \oplus (L, h^L),$$

where (L, h^L) is a trivial Hermitian line bundle. (e.g., E is the holomorphic tangent bundle of the product manifold of a compact Hermitian manifold and a complex torus.) Then, the structure group of E is reducible to $U(r-1) \times 1$. The Hermitian connection ∇ of (E, h) is also splits as

$$\nabla = \nabla' \oplus d,$$

where d is the exterior differentiation on L , that is, the connection form ω of (E, h) with respect to a suitable holomorphic frame field of E is written as $\omega := \begin{pmatrix} \omega' & 0 \\ 0 & 0 \end{pmatrix}$, where ω' is the connection form of (E', h') .

Let $\xi = \xi' + \xi^L$ be the corresponding decomposition of $\xi \in a^0(E)$. Then, we shall define a complex Finsler structure F on E by

$$F(z, \xi) = \frac{1}{2} \left\{ \|\xi\|_E^2 + \sqrt{\|\xi\|_E^4 + 4\|\xi^L\|_{E^L}^4} \right\},$$

where $\|\xi\|_E^2 = h(\xi, \xi)$ and $\|\xi^L\|_{E^L}^2 = h^L(\xi^L, \xi^L)$. The convexity of F is derived by direct calculations.

Let $\xi(t) = \xi'(t) + \xi^L(t) \in a^0(E)$ be a parallel field with respect to ω along a curve $c(t) = (z(t))$. Since $\frac{d}{dt} \|\xi(t)\|_E = \frac{d}{dt} \|\xi^L(t)\|_{E^L} = 0$, we get

$$\frac{d}{dt} F(z(t), \xi(t)) = 0.$$

This means that ω is the partial connection of (E, F) . Thus (E, F) is modeled on a complex Minkowski space, and its associated h_F is the given h . Moreover, (E, F) is Einstein-Finsler if and only if (E, h_F) is Einstein-Hermitian.

We suppose that an Einstein-Finsler bundle (E, F) with a constant factor φ is modeled on a complex Minkowski space. Then, by Theorem 6.2, the partial connection D is given by the Hermitian connection of the associated (E, h_F) . Hence, in this case, all the results of LÜBKE [8] hold, and the constant φ in (5.4) is given by the c in (3.3). Moreover we shall show the following

Theorem 6.3. *Let (E, F) be an Einstein-Finsler bundle over a compact Kähler manifold (M, Φ) . If (E, F) is modeled on a complex Minkowski space, E is Φ -semi-stable and (E, F) is a direct sum*

$$(E, F) = (E_1, F_1) \oplus \cdots \oplus (E_k, F_k),$$

where $F_j := F|_{E_j}$, and each (E_j, F_j) is modeled on a complex Minkowski space whose associated Hermitian vector bundles is a Φ -stable Einstein-Hermitian vector bundle with the same factor c as (E, F) .

PROOF. By Theorem 6.2, if (E, F) is modeled on a complex Minkowski space, then there exists a Hermitian metric h_F of E such that the mean curvature K_j^i in (3.1) is the one of h_F . Therefore (E, h_F) is an Einstein-Hermitian vector bundle over (M, Φ) . So, by Theorem 6.1, E is Φ -semi-stable, and (E, h_F) is a direct sum

$$(E, h_F) = (E_1, h_{F_1}) \oplus \cdots \oplus (E_k, h_{F_k}).$$

Each Finsler bundle (E_j, F_j) , $F_j := F|_{E_j}$ is obviously modeled on a complex Minkowski space, and its associated Hermitian vector bundle (E_j, h_{F_j}) is Φ -stable. $\square$

§6.2. Special case II. Let (E, F) be a convex Finsler bundle. We suppose that *there exists a non-trivial holomorphic ζ section of E such that $D\zeta_P=0$* . For the function $f(z) := F(z, \zeta(z)) = \|\zeta(z)\|^2$, by Proposition 1.1 we have

$$\partial f = (\partial_{\mathcal{H}} F)_{(z, \zeta(z))} = 0,$$

and hence the norm of ζ is constant, and so is a non-vanishing section. This section ζ spans a trivial holomorphic line bundle $L = \langle \zeta \rangle$, and its pull back $\tilde{L}$ is parallel with respect to D . Hence the pull-back $\tilde{E}$ splits partial-holomorphically as

$$(\tilde{E}, H) = (\tilde{E}', H') \oplus (\tilde{L}, H^L),$$

and the partial connection D also splits as

$$D = D' \oplus d_{\mathcal{H}}.$$

Then we say the triplet (E, F, ζ) is *partially reducible*. If (E, F) is Hermitian bundle, this notion is just the reducibility defined in Example 6.1.

Let (E, F, ζ) be a partially reducible convex Finsler bundle. Then we shall define a Hermitian structure $h^\zeta = (h_{i\bar{j}}^\zeta)$ on E by $h^\zeta := f_\zeta^* H$:

$$h_{i\bar{j}}^\zeta(z) := F_{i\bar{j}}(z, [\zeta(z)]).$$

By the discussions below, we can identify the bundle (E, h_ζ) with $(\tilde{E}, H)|_{f_\zeta(M)}$. In fact, the Hermitian connection ∇^ζ of (E, h^ζ) is given by the form $\theta^\zeta = h^{\zeta^{-1}} \partial h^\zeta$. Then, from (4.1) and $D_\alpha \zeta^i = 0$, we get

$$\partial h^\zeta = \partial(f_\zeta^* H) = f_\zeta^* \partial_{\mathcal{H}} H,$$

and so

$$\theta_j^{\zeta i} = \sum_m h^{\zeta \bar{m} i} f_\zeta^* (\partial_{\mathcal{H}} F_{j\bar{m}}) = f_\zeta^* \omega_j^i.$$

The curvature Θ^ζ of (E, h^ζ) is also given by

$$\Theta_j^{\zeta i}(z) = f_\zeta^* \Omega_j^i = \sum_{\alpha, \beta} R_{j\alpha\bar{\beta}}^i(z, [\zeta(z)]) dz^\alpha \wedge d\bar{z}^\beta,$$

and its mean curvature K^ζ by $K_j^{\zeta i}(z) = K_j^i(z, [\zeta(z)])$.

Then, if (E, F) is an Einstein-Finsler bundle with constant factor φ , the bundle (E, h_ζ) is Einstein-Hermitian with the factor φ . Since the degree of E is independent on the choice of Hermitian structure, the constant φ in (5.4) is given by c in (3.3). Thus, from Theorem 6.1 we get

Theorem 6.4. *Let (E, F, ζ) be a partially reducible Finsler bundle over a compact Kähler manifold (M, Φ) . If (E, F) satisfies the Einstein-Finsler condition, then E is Φ -semi-stable, and (E, h_ζ) is a direct sum*

$$(E, h_\zeta) = (E_1, h_{\zeta_1}) \oplus \cdots \oplus (E_k, h_{\zeta_k}),$$

where (E_j, h_{ζ_j}) is a Φ -stable Einstein-Hermitian vector bundle with the same factor c as (E, F) .

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(Received April 29, 1997)