

On Finsler spaces of Douglas type A generalization of the notion of Berwald space

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Dedicated to Professor Lajos Tamássy on his 75th birthday

Abstract. A new generalization of the notion of Berwald space is proposed from the viewpoint of the equation of geodesics. A Douglas space is characterized by the vanishing Douglas tensor. Various examples of Douglas spaces are given in relation to other special Finsler spaces.

1. Introduction

We consider a geodesic curve $C : x^i = x^i(t)$, $t_0 \leq t \leq t_1$, of an n -dimensional Finsler space $F^n = (M^n, L(x, y))$ on a smooth n -manifold M^n , equipped with the fundamental function $L(x, y)$, $x = (x^i)$, $y = (y^i)$. C is the extremal of the length integral $s = \int_{t_0}^{t_1} L(x, \dot{x}) dt$, $\dot{x}^i = dx^i/dt$, given by the Euler equation

$$(1.1) \quad E_i(C) = \frac{d}{dt} L_{(i)} - L_i = 0,$$

where $L_{(i)} = \dot{\partial}_i L$ and $L_i = \partial_i L$. Putting $F = L^2/2$, we get the fundamental tensor $g_{ij} = \dot{\partial}_j \dot{\partial}_i F$ and the well-known functions

$$2G_j = (\dot{\partial}_j \partial_r F) y^r - \partial_j F.$$

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Then, $(g^{ij}) = (g_{ij})^{-1}$ and $G^i = g^{ij}G_j$, we get

$$Lg^{ij}E_j(C) = \ddot{x}^i + 2G^i(x, \dot{x}) - \frac{\ddot{s}}{\dot{s}}\dot{x}^i = 0.$$

Consequently C is given by the system of differential equations

$$(1.2) \quad \ddot{x}^i \dot{x}^j - \ddot{x}^j \dot{x}^i + 2D^{ij}(x, \dot{x}) = 0,$$

where we put

$$(1.3) \quad D^{ij}(x, y) = G^i(x, y)y^j - G^j(x, y)y^i.$$

We are, in particular, concerned with a two-dimensional Finsler space F^2 with a local coordinate system $(x^1, x^2) = (x, y)$ and we put $(y^1, y^2) = (p, q)$. Let us take x as a parameter of curves and denote $y' = dy/dx = q/p$ and $y'' = d^2y/dx^2 = (p\dot{q} - \dot{p}q)/p^3$. Since $D^{ij}(x, y; p, q)$ are positively homogeneous in (p, q) of degree three, we have $D^{ij}(x, y; p, q) = p^3 D^{ij}(x, y; 1, q/p)$, provided $p > 0$. Consequently (1.2) can be written in the form

$$y'' = 2\{G^1(x, y; 1, y')y' - G^2(x, y; 1, y')\}.$$

We consider the Berwald connection $B\Gamma = (G_i{}^j{}_k, G_j^i)$ ([1], [13]), which is given by $G_j^i = \dot{\partial}_j G^i$ and $G_j{}^i{}_k = \dot{\partial}_k G_j^i$. Then we get $2G^i = G_j{}^i{}_k y^j y^k$, and the equation above can be written in the form

$$(1.4) \quad y'' = X_3(y')^3 + X_2(y')^2 + X_1 y' + X_0,$$

where we put

$$(1.4a) \quad \begin{aligned} X_3 &= G_2{}^1{}_2, & X_2 &= 2G_1{}^1{}_2 - G_2{}^2{}_2, \\ X_1 &= G_1{}^1{}_1 - 2G_1{}^2{}_2, & X_0 &= -G_1{}^2{}_1. \end{aligned}$$

Suppose that the F^2 under consideration is a Berwald space ([1], [13]), that is, $G_j{}^i{}_k$ are functions of position (x, y) alone. Then the X 's of (1.4a) are functions of (x, y) and, in consequence, (1.4) shows that the right-hand side of the equation $y'' = f(x, y, y')$ of a geodesic is a polynomial in y' of degree at most three. If our discussion is restricted to Riemannian space of dimension two, then $G_j{}^i{}_k$ are Christoffel symbols, and hence $f(x, y, y')$ is, of course, a polynomial in y' of degree at most three for all geodesics of any two-dimensional Riemannian space.

The remarkable property of $y'' = f(x, y, y')$ as above given does not depend on the choice of coordinates (x, y) . In fact, we have the following

Lemma. We consider an ordinary differential equation of second order having the following form

$$(1.5) \quad y'' = Y_3(y')^3 + Y_2(y')^2 + Y_1y' + Y_0,$$

where the Y 's are functions of (x, y) . This special form is preserved under any transformation of variables.

PROOF. Suppose that we have a differential equation

$$\bar{y}'' = \bar{Y}_3(\bar{y}')^3 + \bar{Y}_2(\bar{y}')^2 + \bar{Y}_1\bar{y}' + \bar{Y}_0,$$

where $\bar{y}' = d\bar{y}/d\bar{x}$ and the $\bar{Y}$'s are functions of $(\bar{x}, \bar{y})$. Let us consider a transformation $(\bar{x}, \bar{y}) \rightarrow (x, y)$, given by $\bar{x} = f(x, y)$ and $\bar{y} = g(x, y)$. Then it is easy to find the transformed equation from (1.5) with the Y 's as coefficients as follows: Putting $J = f_x g_y - f_y g_x$, we obtain

$$\begin{aligned} JY_3 &= g_y^3 \bar{Y}_3 + g_y^2 f_y \bar{Y}_2 + g_y f_y^2 \bar{Y}_1 + f_y^3 \bar{Y}_0 - f_y g_{yy} + f_{yy} g_y, \\ JY_2 &= 3g_x g_y^2 \bar{Y}_3 + (f_x g_y + 2f_y g_x) g_y \bar{Y}_2 + (f_y g_x + 2f_x g_y) f_y \bar{Y}_1 \\ &\quad + 3f_x f_y^2 \bar{Y}_0 - f_x g_{yy} + g_x f_{yy} + 2(f_{xy} g_y - g_{xy} f_y), \\ JY_1 &= 3g_x^2 g_y \bar{Y}_3 + (f_y g_x + 2f_x g_y) g_x \bar{Y}_2 + (f_x g_y + 2f_y g_x) f_x \bar{Y}_1 \\ &\quad + 3f_x^2 f_y \bar{Y}_0 - f_y g_{xx} + g_y f_{xx} + 2(f_{xy} g_x - g_{xy} f_x), \\ JY_0 &= g_x^3 \bar{Y}_3 + g_x^2 f_x \bar{Y}_2 + g_x f_x^2 \bar{Y}_1 + f_x^3 \bar{Y}_0 - f_x g_{xx} + f_{xx} g_x. \end{aligned}$$

It is observed from the above that " $Y_3 = 0$ ", for instance, is not preserved by such a transformation of variables.

The following proposition is obvious from the definition (1.3) and the homogeneity of D^{ij} :

Proposition 1. The right-hand side of the equation (1.4) is a polynomial in y' of degree at most three, if and only if $D^{12}(x, y; p, q)$ is a homogeneous polynomial in (p, q) of degree three.

The Lemma suggests that the differential equation of the type (1.5) will also be of great value and interest from a geometrical point of view. In fact, E. CARTAN ([7], p. 242) defines a projective connection and constructs two-dimensional differential-geometric entities.

2. Douglas space

Definition. A Finsler space is said to be of *Douglas type* or a *Douglas space*, if $D^{ij} = G^i y^j - G^j y^i$ are homogeneous polynomials in (y^i) of degree three.

Theorem 1. A two-dimensional Finsler space is a Douglas space if and only if, in a local coordinate system (x, y) , the right-hand side $f(x, y, y')$ of the equation of geodesics $y'' = f(x, y, y')$ is a polynomial in y' of degree at most three.

We treat of a Finsler space F^n with the Berwald connection $B\Gamma = (G_j^i, G_j^i)$. F^n is by definition a Douglas space if and only if

$$\dot{\partial}_k \dot{\partial}_j \dot{\partial}_i \dot{\partial}_h (G^l y^m - G^m y^l) = 0.$$

We have first for $D^{lm} = G^l y^m - G^m y^l$

$$\dot{\partial}_h D^{lm} = G_h^l y^m + G^l \delta_h^m - [l, m],$$

where $[l, m]$ denotes the interchange of indices (l, m) of the preceding terms. Next

$$\begin{aligned} \dot{\partial}_i \dot{\partial}_h D^{lm} &= G_h^l y^m + G_h^l \delta_i^m + G_i^l \delta_h^m - [l, m], \\ \dot{\partial}_j \dot{\partial}_i \dot{\partial}_h D^{lm} &= G_h^l y^m + \{G_h^l \delta_j^m + (h, i, j)\} - [l, m], \end{aligned}$$

where (h, i, j) denotes the cyclic permutation of the indices (h, i, j) of the preceding terms in the parentheses and $G_h^l \delta_{ij} = \dot{\partial}_j G_h^l = \dot{\partial}_j \dot{\partial}_i \dot{\partial}_h G^l$ are the components of the *hv-curvature tensor* of $B\Gamma$ ([1], p. 86; [13], p. 118). Further, introducing the tensor $G_h^l \delta_{ijk} = \dot{\partial}_k G_h^l \delta_{ij}$, we obtain

$$(2.1) \quad \dot{\partial}_k \dot{\partial}_j \dot{\partial}_i \dot{\partial}_h D^{lm} (= D_{hijk}^{lm}) = G_h^l y^m + \{G_h^l \delta_{ij}^m + (h, i, j, k)\} - [l, m],$$

where (h, i, j, k) is the symbol analogous to (h, i, j) . D_{hijk}^{lm} are components of a tensor and $D_{hijk}^{lm} = 0$ is necessary and sufficient for F^n to be a Douglas space. By $G_h^l \delta_{ijr} y^r = -G_h^l \delta_{ij}$, (2.1) yields

$$(2.2) \quad D_{hijr}^{lr} = (n+1) D_h^l \delta_{ij},$$

where $D_h^l{}_{ij}$ are components of the well-known *Douglas tensor* ([1], (3.3.2.7); [4]; [11]):

$$(2.3) \quad D_h^l{}_{ij} = G_h^l{}_{ij} - \frac{1}{n+1} G_{hij} y^l - \frac{1}{n+1} \{G_{hi} \delta_j^l + (h, i, j)\},$$

where $G_{hi} = G_h^r{}_{ir}$ is the *hv-Ricci tensor* of $B\Gamma$ and $G_{hij} = \dot{\partial}_j G_{hi} = G_h^r{}_{irj}$. Therefore the Douglas tensor must vanish for F^n .

Conversely, if the Douglas tensor of an F^n vanishes identically, then F^n is a Douglas space, because it is easy to show the following equality:

$$(2.4) \quad D_{hijk}^{lm} = (\dot{\partial}_k D_h^l{}_{ij}) y^m + \{D_{ijk}^l \delta_h^m + (h, i, j, k)\} - [l, m].$$

Therefore we have the following

Theorem 2. *A Finsler space is of Douglas type, if and only if the Douglas tensor vanishes identically.*

Historical remark. It is well-known that the Douglas tensor (2.3) has been introduced first by J. DOUGLAS in 1928 in his paper on the general geometry of paths [8]. He used the German letter $\mathfrak{H}$ to show this tensor ([8], (5.10), (5.11)), because the Roman letter H was already used to show the *hv*-curvature tensor G . In BERWALD's paper [5], published in 1941, he proved that a two-dimensional Landsberg space with vanishing Douglas tensor is a Berwald space ([5], p. 110). In one of his posthumous papers [6], published in 1947, he used the letter D to denote the Douglas tensor which is preserved invariant under projective change, and showed that $D = 0$ is one half of the necessary and sufficient conditions for a generalized affine space to be projectively flat. The Douglas tensor also appeared in H . Rund's monograph ([20], p. 143), denoted by the letter B .

In 1980 Z. I. SZABÓ's paper [21] on the global foundations of Finsler projective geometry was published; he denoted the Douglas tensor by the letter D ([21], (3.3), (3.11)) and proposed first the name "Douglas tensor". Almost simultaneously the second author of the present paper was concerned with this tensor ([11], (2.10)) and called it the projective *hv*-curvature tensor or the Douglas tensor.

Recently the first author of the present paper considered $n(>2)$ -dimensional Landsberg spaces with vanishing Douglas tensor, and proved an extension of Berwald's theorem as above. This theorem was supplemented and completed by the present authors [4].

Using the name "Douglas space", this theorem can be stated as follows:

Theorem 3. *If a Finsler space F^n ($n \geq 2$) is a Landsberg space and a Douglas space, then it is a Berwald space. Conversely a Berwald space is a Landsberg space and a Douglas space.*

As is well-known, the Douglas tensor is projectively invariant. Hence we have the

Theorem 4. *If a Finsler space is projectively related to a Douglas space, then it is also a Douglas space.*

Example 1. In a previous paper of the second author [16] it is shown that the family of solutions of a second order linear differential equation

$$y'' + P(x)y' + Q(x)y = R(x)$$

coincides with the family of geodesics of the two-dimensional Finsler space F^2 with the metric

$$L(x, y; p, q) = \frac{1}{p} e^{\int P dx} [(2R - Qy)yp^2 + q^2] + E_x p + E_y q,$$

where $E = E(x, y)$ is an arbitrary function. Consequently this F^2 is a Douglas space. It is observed that this metric is of Kropina type (cf. Theorem 8).

3. Douglas spaces with projective connection

It seems that the Douglas tensor has made a contribution only to the theory of projective changes.

A Finsler space F^n is said to be projectively related or projective to another Finsler space $\bar{F}^n$, if any geodesic of F^n is a geodesic of $\bar{F}^n$ and vice versa. The condition for it is written as

$$(3.1) \quad \bar{G}^i = G^i + P y^i,$$

where $P = P(x, y)$ is a scalar function. The change $F^n \rightarrow \bar{F}^n$ is called projective.

From (3.1) we have

$$(3.2) \quad \bar{G}_i^h = G_i^h + P_i y^h + P \delta_i^h,$$

$$(3.3) \quad \bar{G}_i^{h_j} = G_i^{h_j} + P_{ij} y^h + P_i \delta_j^h + P_j \delta_i^h,$$

where $P_i = \dot{\partial}_i P$ and $P_{ij} = \dot{\partial}_j P_i$. If we put $G = G_r^r$, then (3.1) and (3.2) give

$$\bar{G} = G + (n+1)P, \quad \bar{G}^h - \frac{\bar{G}}{n+1}y^h = G^h - \frac{G}{n+1}y^h,$$

which gives rise to a projective invariant

$$(3.4) \quad Q^h = G^h - \frac{G}{n+1}y^h, \quad G = G_r^r.$$

Putting $Q_i^h = \dot{\partial}_i Q^h$ and $Q_i{}^h{}_j = \dot{\partial}_j Q_i^h$, we have

$$(3.5) \quad Q_i{}^h{}_j = G_i{}^h{}_j - \frac{1}{n+1}(G_{ij}y^h + G_r{}^r{}_i\delta_j^h + G_r{}^r{}_j\delta_i^h).$$

Finally we get a remarkable expression of the Douglas tensor as follows:

$$(3.6) \quad \dot{\partial}_k \dot{\partial}_j \dot{\partial}_i Q^h = D_i{}^h{}_{jk}.$$

Remark. Though K. YANO's notation in his monograph [23] is quite different from our notation, his (9.28), p. 197 is just the same with our (3.6). See [11], (5.1).

We are led from (3.6) and Theorem 2 to the

Theorem 5. *A Finsler space is of Douglas type, if and only if Q^h of (3.4) are homogeneous polynomials in (y^i) of degree two.*

Let F^n be a Douglas space. Then $Q_i{}^h{}_j$ of (3.5) are functions of position (x) alone and we may put

$$Q_i{}^h{}_j(x)y^i y^j = 2G^h - \frac{2}{n+1}G_r^r y^h,$$

which implies

$$(3.7) \quad 2D^{hk} = (Q_i{}^h{}_j(x)y^i y^j)y^k - [h, k].$$

Proposition 2. *For a Douglas space, $D^{hk} = G^h y^k - G^k y^h$ are homogeneous polynomials in (y^i) of degree three as written in the form (3.7).*

In particular, for a two-dimensional Douglas space, the equation of geodesics $y'' = f(x, y, y')$ is written as (1.5). As it is seen from (1.4), (1.5)

is written in more convenient form as

$$(3.8) \quad y'' = Q_2^1{}_2 (y')^3 + (2Q_1^1{}_2 - Q_2^2{}_2)(y')^2 + (Q_1^1{}_1 - 2Q_1^2{}_2)y' - Q_1^2{}_1,$$

where $Q_j^i{}_k = Q_j^i{}_k(x)$.

We consider a Finsler space F^n with the *projective connection* $P\Gamma = (P_j^i{}_k, G_j^i)$ [14], where the h -connection coefficients $P_j^i{}_k$ are given by

$$(3.9) \quad P_j^i{}_k = G_j^i{}_k - \frac{1}{n+1} G_{jk} y^i.$$

Since $G_{jk} = G_j^r{}_{kr}$ is a tensor, called the hv -Ricci tensor, $(P_j^i{}_k)$ define certainly a horizontal connection ([13], §9). It is noteworthy that $P_j^i{}_k$ appear in (3.5):

$$(3.5') \quad Q_j^i{}_k = P_j^i{}_k - \frac{1}{n+1} (G_r^r{}_j \delta_k^i + G_r^r{}_k \delta_j^i).$$

Remark. In Yano's monograph [23] the $P_j^i{}_k$ are denoted by the Greek letter Π ((9.11), p. 196), and called the connection coefficients of the normal projective connection. On the other hand, the name "projective connection coefficients" appears in Rund's monograph [20], p. 142, and coincides with $Q_i^h{}_j$ of (3.5). Rund writes:

They define a projective covariant derivative in the same manner as the $G_j^i{}_k$ give rise to a covariant derivative. However the projective covariant derivatives of a tensor are not, in general, tensors.

Thus it is sure that Rund did not recognize the entity $(Q_j^i{}_k)$ as a connection in the modern sense; nevertheless, how many strange papers have been devoted to studying Rund's theory of projective connection!

According to the theory of the paper [14], our $P\Gamma$ is symmetric and L -metrical: $L_{;i} = 0$. Its deflection tensor $y^r P_r^i{}_k - G_k^i$ vanishes and the $(v)hv$ -torsion tensor $U_j^i{}_k = \dot{\partial}_k G_j^i - P_k^i{}_j$ is given by

$$(3.10) \quad U_j^i{}_k = \frac{1}{n+1} y^i G_{jk}.$$

The hv -curvature tensor of $P\Gamma$, denoted by $U_i^h{}_{jk}$, is equal to $\dot{\partial}_k P_i^h{}_j$, which is written as

$$(3.11) \quad U_i^h{}_{jk} = G_i^h{}_{jk} - \frac{1}{n+1} (G_{ijk} y^h + G_{ij} \delta_k^h).$$

Since the hv -Ricci tensor is $U_{ij} = U_i^r{}_{jr} = 2G_{ij}/(n+1)$, (2.3) and (3.11) lead to the expression of the Douglas tensor in terms of $P\Gamma$ as follows:

$$(3.12) \quad D_i^h{}_{jk} = U_i^h{}_{jk} - \frac{1}{2}(\delta_i^h U_{jk} + \delta_j^h U_{ik}).$$

Consequently we have the

Theorem 6. *In terms of the projective connection $P\Gamma$ a Douglas space is characterized by the equation*

$$U_i^h{}_{jk} = \frac{1}{2}(\delta_i^h U_{jk} + \delta_j^h U_{ik}),$$

where $U_i^h{}_{jk}$ and U_{ij} are the hv -curvature tensor and the hv -Ricci tensor.

Remark. On p. 244 of Cartan's monograph [7] we find the differential equation

$$\begin{aligned} \frac{d^2 v}{du^2} = & \Pi_2^1{}_2 \left(\frac{dv}{du} \right)^3 + (2\Pi_1^1{}_2 + \Pi_2^1{}_1) \left(\frac{dv}{du} \right)^2 \\ & + (2\Pi_1^1{}_1 - \Pi_1^2{}_2) \frac{dv}{du} - \Pi_1^2{}_1, \end{aligned}$$

where $\Pi_j^i{}_k = \Pi_j^i{}_k(u, v)$ are coefficients of Cartan's projective connection. It is seemingly analogous to (3.8), but $Q_j^i{}_k$ of (3.8) are not the projective connection $P\Gamma$; however $P_j^i{}_k$ may depend on (y^i) of (x^i, y^i) .

4. Wagner spaces of Douglas type

The notion of Wagner space was originally defined by V.V. WAGNER in 1943 [22] and established strictly from the modern standpoint by M. HASHIGUCHI in 1975 [9] (cf. [2]; [13], Definition 25.4).

Let $s_i(x)$ be components of a covariant vector field on an n -manifold M . Then the *Wagner connection* $W\Gamma(s) = (F_j^i{}_k, N_j^i, C_j^i{}_k)$ of a Finsler space $F^n = (M^n, L(x, y))$ is by definition a Finsler connection which is uniquely determined by the following five axioms:

- (1) h -metrical: $g_{ij|k} = 0$,
- (2) $(h)h$ -torsion tensor $T_j^i{}_k = F_j^i{}_k - F_k^i{}_j$ is given by $T_j^i{}_k = \delta_j^i s_k - \delta_k^i s_j$.
- (3) deflection tensor: $D_j^i = y^r F_r^i{}_j - N_j^i = 0$,
- (4) v -metrical: $g_{ij|k} = 0$,
- (5) $(v)v$ -torsion tensor: $S_j^i{}_k = C_j^i{}_k - C_k^i{}_j = 0$.

Thus $WT(s)$ is not intrinsically defined in F^n , but is a geometrical structure on M^n given by $L(x, y)$ together with $s_i(x)$.

A Finsler space F^n is called a *Wagner space*, if its $WT(s)$ is linear, that is, $F_j^{i_k}$ are functions of position (x^i) alone. Consequently the notion of Wagner space is regarded as a generalization of Berwald space.

Let $CT = (\Gamma_{jk}^{*i}, G_j^i, C_j^{i_k})$ be the Cartan connection of a Finsler space with a Wagner connection $WT(s)$. The difference $D_j^{i_k} = F_j^{i_k} - \Gamma_{jk}^{*i}$ of $WT(s)$ from CT is given [2] by

$$(4.1) \quad D_j^{i_k} = V_j^{i_{kr}} s^r, \quad s^r = g^{ri} s_i(x),$$

$$(4.1a) \quad V_j^{i_{kh}} = g_{jk} \delta_h^i - g_{jh} \delta_k^i - C_j^{i_h} y_k - C_k^{i_h} y_j + C_j^{i_k} y_h \\ + C_{jkh} y^i + L^2 (S_j^{i_{kh}} + C_j^{i_r} C_k^{r_h}),$$

where $y_i = g_{ir} y^r = LL_{(i)}$ and $S_j^{i_{kh}}$ is the v -curvature tensor of CT . Consequently we have

$$V_0^{i_{0h}} = L^2 h_h^i, \quad D_0^{i_0} = L^2 s^i - s_0 y^i,$$

where $h_h^i = \delta_h^i - l^i l_h$ is the angular metric tensor. Thus we get

$$(4.2) \quad F_0^{i_0} = 2G^i + L^2 s^i - s_0 y^i,$$

which implies

$$F_0^{i_0} y^j - F_0^{j_0} y^i = 2(G^i y^j - G^j y^i) + L^2 (g^{ir} y^j - g^{jr} y^i) s_r.$$

Therefore from the definition of Wagner space we obtain

Proposition 3. *For a Wagner space with $WT(s)$,*

$$2(G^i y^j - G^j y^i) + L^2 (g^{ir} y^j - g^{jr} y^i) s_r$$

are homogeneous polynomials in (y^i) of degree three.

From the definition of Douglas type and Proposition 3 it follows

Theorem 7. *Let F^n be a Wagner space with a Wagner connection $WT(s)$. F^n is of Douglas type, if and only if*

$$W^{ij} = L^2 (g^{ir} y^j - g^{jr} y^i) s_r$$

are homogeneous polynomials in (y^i) of degree three.

Example 2. We consider a Kropina space $F^n = (M^n, L = \alpha^2/\beta)$, $\alpha^2 = a_{ij}(x)y^i y^j$, $\beta = b_i(x)y^i$ ([13], p. 107). The fundamental tensor g_{ij} and g^{ij} of F^n are given on account of the formulae (30.4) and (30.7) of [13] as follows:

$$g_{ij} = \frac{2\alpha^2}{\beta^2} a_{ij} + \frac{3\alpha^4}{\beta^4} b_i b_j - \frac{4\alpha^2}{\beta^3} (b_i Y_j + b_j Y_i) + \frac{4}{\beta^2} Y_i Y_j,$$

$$g^{ij} = \frac{\beta^2}{2\alpha^2} a^{ij} - \frac{\beta^2}{2b^2 \alpha^2} B^i B^j + \frac{\beta^3}{b^2 \alpha^2} (B^i y^j + B^j y^i) + \frac{\beta^2}{\alpha^4} \left(1 - \frac{2\beta^2}{b^2 \alpha^2}\right) y^i y^j,$$

where $Y_i = a_{ir} y^r$, $B^i = a^{ir} b_r$ and $b^2 = b_r b^r$. Consequently we get

$$W^{ij} = \frac{1}{2b^2} (b^2 \alpha^2 a^{ir} - \alpha^2 B^i B^r + 2\beta B^i y^r) s_r y^j - [i, j].$$

It is obvious that W^{ij} above are homogeneous polynomials in (y^i) of degree three. Therefore F^n is of Douglas type, provided that it is a Wagner space. In particular, a Kropina space F^2 is a Wagner space, as shown by the second author of the present paper [12] (cf. [2], [19]). Therefore we have the

Theorem 8. *Let F^n be a Kropina space. (1) If F^n ($n > 2$) is a Wagner space, then it is a Douglas space. (2) F^2 is a Douglas space.*

Remark. The second author is so sorry to correct the equation (30.7') of his monograph [13] as follows:

$$s_{-1} = \frac{1}{\tau p} \{pp_{-1} - (p_0 p_{-2} - p_{-1}^2) \beta\}.$$

We shall continue to consider Wagner spaces of Douglas type, according to Theorem 5. For a Wagner space F^n with $W\Gamma(s)$ we have from (4.2)

$$F_j^i{}_0 + F_0^i{}_j = 2G_j^i + 2Ls^i l_j - 2L^2 C_j^i{}_r s^r - y^i s_j - s_0 \delta_j^i.$$

The second axiom $F_j^i{}_k - F_h^i{}_j = \delta_j^i s_k - \delta_k^i s_j$ yields

$$F_0^i{}_j = G_j^i - L^2 C_j^i{}_r s^r - s_0 \delta_j^i + s^i y_j,$$

which implies

$$F_0^r{}_r = G_r^r - L^2 C^r s_r - (n-1)s_0, \quad C^r = g^{ij} C_i^r{}_j.$$

Consequently Q^i of (3.4) is written as

$$(4.3) \quad Q^i = \left\{ \frac{1}{2} F_0^i{}_0 - \frac{1}{n+1} F_0^r{}_r y^i + \left(\frac{1}{2} - \frac{n-1}{n+1} \right) s_0 y^i \right\} \\ + L^2 \left(\frac{1}{2} s^i + \frac{1}{n+1} C^r s_r y^i \right).$$

The terms in the first parentheses of (4.3) are homogeneous polynomials in (y^i) of degree two, and hence we have

Theorem 9. *Let F^n be a Wagner space with a Wagner connection $WT(s)$. F^n is a Douglas space, if and only if*

$$V^i = L^2 \{ (n+1) g^{ir} + 2y^i C^r \} s_r$$

are homogeneous polynomials in (y^i) of degree two.

Similarly as in the case of (3.7), W^{ij} of Theorem 7 are in relation to V^i as

$$(4.4) \quad (n+1) W^{ij} = V^i y^j - V^j y^i.$$

5. Two-dimensional Douglas spaces

We consider two-dimensional Douglas spaces based on the Berwald frame and the main scalar $I(x, y)$ ([13], §28; [1], 3.5; [4]).

In a two-dimensional Finsler space F^2 we have an orthonormal frame field, called the Berwald frame (l, m) ; the vector fields are defined by

$$(5.1) \quad \begin{cases} l^i = \frac{1}{L} y^i, & l_i = L_{(i)}, & h_{ij} = \varepsilon m_i m_j, & \varepsilon = \pm 1, \\ l_i m^i = 0, & m_i m^i = \varepsilon, \end{cases}$$

where h_{ij} is the angular metric tensor $h_{ij} = L L_{(i)(j)}$ and the sign ε is the signature of F^2 . Then we get

$$(5.2) \quad \begin{cases} g_{ij} = l_i l_j + \varepsilon m_i m_j, \\ (m_i) = h(-l^2, l^1), & (m^i) = k(-l_2, l_1), & hk = \varepsilon, \\ g(= \det(g_{ij})) = \varepsilon h^2. \end{cases}$$

As is shown in [17], we have

$$(5.3) \quad 2G^i = L_0 l^i + \frac{L^2 M}{h} m^i,$$

$$(5.3a) \quad L_0 = L_r y^r, \quad M = L_{1(2)} - L_{2(1)},$$

where $L_r = \partial_r L$ and $L_{(i)} = \dot{\partial}_i L$.

For an F^2 we can introduce the Weierstrass invariant

$$(5.4) \quad W = \frac{L_{(1)(1)}}{(y^2)^2} = \frac{-L_{(1)(2)}}{y^1 y^2} = \frac{L_{(2)(2)}}{(y^1)^2}.$$

Hence we have

$$h_{11} = LL_{(1)(1)} = LW(y^2)^2, \quad h_{11} = \varepsilon(m_1)^2 = \varepsilon h^2 (l^2)^2,$$

which implies

$$(5.5) \quad L^3 W = \varepsilon h^2 = g.$$

Consequently (5.3) is rewritten in a simpler form as

$$(5.3') \quad 2G^1 = \frac{1}{L} \left(L_0 y^1 - \frac{M}{W} L_{(2)} \right), \quad 2G^2 = \frac{1}{L} \left(L_0 y^2 + \frac{M}{W} L_{(1)} \right).$$

Therefore D^{12} of (1.3) is written in the remarkable form

$$(5.6) \quad 2D^{12} = -\frac{1}{W} (L_{1(2)} - L_{2(1)}).$$

Theorem 10. *A two-dimensional Finsler space is a Douglas space, if and only if $(L_{1(2)} - L_{2(1)})/W$ is a homogeneous polynomial in (y^1, y^2) of degree three, where W is the Weierstrass invariant.*

We have, particularly in an F^2 , a simple form of the equation of geodesics, called the *Weierstrass form* ([1], (1.1.3.2); [16], p. 296; [17], (1.4)):

$$(5.7) \quad p\dot{q} - \dot{p}q + \frac{1}{W} (L_{xq} - L_{yp}) = 0,$$

where $(x^i) = (x, y)$ and $(y^i) = (p, q)$. Thus Theorem 10 is shown directly from the definition of Douglas space.

Example 3. We consider a two-dimensional Finsler space with the metric

$$L(x, y; p, q) = q \tan^{-1} \frac{q}{p} - p \log \sqrt{1 + \left(\frac{q}{p}\right)^2} - xq.$$

The differential equation of the geodesics of F^2 is given by

$$y'' = (y')^2 + 1,$$

which shows that F^2 is a Douglas space ([16], Example 4). The finite equation of geodesics is

$$y = c_1 - \log |\cos(x + c_2)|,$$

where the c 's are arbitrary constants. We have

$$L_{xq} - L_{yp} = -1, \quad L_0 = -pq, \quad L_{pp} = \frac{q^2}{p(p^2 + q^2)}, \quad W = \frac{1}{p(p^2 + q^2)}.$$

Thus (5.3') leads to

$$\begin{aligned} 2LG^1 &= -p^2q + p(p^2 + q^2) \left(\tan^{-1} \frac{q}{p} - x \right), \\ 2LG^2 &= -pq^2 + p(p^2 + q^2) \log \sqrt{1 + \left(\frac{q}{p}\right)^2}. \end{aligned}$$

Consequently F^2 is certainly not a Berwald space, but we have $2(G^1q - G^2p) = p(p^2 + q^2)$, which implies again that F^2 is a Douglas space.

The *main scalar* $I(x, y; p, q)$ of F^2 is a scalar, positively homogeneous in (p, q) of degree zero, defined from the C -tensor as

$$LC_{ijk} = Im_i m_j m_k.$$

Then the hv -curvature tensor $G_i^h{}_{jk}$ of $B\Gamma$ is written ([13], §28; [1], 3.5; [4]) as

$$LG_i^h{}_{jk} = (-2I_{,1}l^h + I_2m^h)m_im_jm_k, \quad I_2 = I_{,1;2} + I_{,2}.$$

The scalar derivatives $(S_{,1}, S_{,2})$ and $(S_{;1}, S_{;2})$ of a scalar field S are defined by

$$S|_i = S_{,1}l_i + S_{,2}m_i, \quad LS|_i = S_{;1}l_i + S_{;2}m_i,$$

where $S|_i = \partial_i S - (\dot{\partial}_r S)G_i^r$ and $S|_i = \dot{\partial}_i S$. We have $S_{;1} = 0$ if S is of degree zero in (p, q) . Then (2.3) leads to the expression of the Douglas tensor as follows:

$$(5.8) \quad 3LD_i{}^h{}_{jk} = -(6I_{,1} + \varepsilon I_{2;2} + 2II_2)m_i l^h m_j m_k.$$

Theorem 11. *A two-dimensional Finsler space is a Douglas space, if and only if the main scalar I satisfies the equation*

$$D^2 = 6I_{,1} + \varepsilon I_{2;2} + 2II_2 = 0, \quad I_2 = I_{,1;2} + I_{,2}.$$

We first deal with a Douglas space F^2 with vanishing T -tensor. This tensor is defined by

$$T_{hijk} = LC_{hij}|_k + l_h C_{ijk} + l_i C_{jkh} + l_j C_{khi} + l_k C_{hij},$$

where $|_k$ denotes the v -covariant differentiation in CT . In terms of the Berwald frame we have ([13], §28; [1], 3.5.3.)

$$LT_{hijk} = I_{;2}m_h m_i m_j m_k.$$

Consequently the conditions for F^2 under consideration are as follows:

$$(5.9) \quad (1) \quad D^2 = 6I_{,1} + \varepsilon I_{2;2} + 2II_2 = 0, \quad (2) \quad I_{;2} = 0.$$

We have to pay attention to two of the Ricci identities for the scalar derivatives;

$$(5.10) \quad \begin{aligned} (1) \quad & S_{,1;2} - S_{,2;1} = S_{,2}, \\ (2) \quad & S_{,2;2} - S_{;2,2} = -\varepsilon(S_{,1} + IS_{,2} + I_{,2}S_{;2}). \end{aligned}$$

Consequently we observe for F^2 that

$$I_{,1;2} = I_{,2}, \quad I_{,2;2} = -\varepsilon(I_{,1} + II_{,2}).$$

Hence we get $I_2 = 2I_{,2}$ and (1) of (5.9) is reduced to

$$(5.11) \quad 2I_{,1} + II_{,2} = 0.$$

Differentiating (;) we get from (5.11) an equation, which is rewritten as

$$-\varepsilon II_{,1} + (2 - \varepsilon I^2)I_{,2} = 0.$$

This together with (5.11) yields $4 - \varepsilon I^2 = 0$, if $I_{,1} = I_{,2} = 0$ do not hold. Even if they hold, we get $I = \text{const}$ from (2) of (5.9). Consequently I is reduced to a constant, and hence F^2 is a Berwald space ([1], Theorem 3.5.3.1; [13]).

Theorem 12. *If a two-dimensional Douglas space has vanishing T-tensor, then it is a Berwald space with constant main scalar.*

Next we consider Wagner spaces F^2 of dimension two. In terms of the Berwald frame, W^{ij} of Theorem 7 is written as

$$W^{ij} = L^3 \varepsilon (m^i l^j - m^j l^i) m^r s_r.$$

Putting $s_i = s_1 l_i + s_2 m_i$, we get $m^r s_r = \varepsilon s_2$, and hence $W^{12} = L^3 (m^1 l^2 - m^2 l^1) s_2 = -L^3 k s_2$, $k^2 = \varepsilon/g$. Therefore we have from Theorem 7 the following

Theorem 13. *Let F^2 be a two-dimensional Wagner space with $W\Gamma(s)$. F^2 is a Douglas space, if and only if $(L^3/\sqrt{|g|})s_2$ is a homogeneous polynomial in (y^1, y^2) of degree three, where $s_i = s_1 l_i + s_2 m_i$.*

Remark. This is an interesting result, compared with the following fact from the theory of Berwald spaces of dimension two ([5]; [13], p. 189; [1], p. 139); $I_{,2} = 0$, that is, I is a function of position alone, if and only if $L^2/\sqrt{|g|}$ is a homogeneous polynomial in (y^1, y^2) of degree two.

We have a remarkable theorem on two-dimensional Wagner spaces, shown by Wagner ([22], [12]): F^2 is a Wagner space, if and only if $I_{,2}$ can be written as a function $f(I)$ of I , provided that $I_{,2} \neq 0$, that is, the T-tensor $\neq 0$. Since Theorem 12 has been shown, we may consider only two-dimensional Wagner space with $I_{,2} \neq 0$, and hence $I_{,2} = f(I)$ may be supposed in the following.

Now we deal with a Wagner space F^2 of Douglas type with $I_{,2} = f(I) \neq 0$. Then (5.10) shows that

$$I_{,1;2} = f' I_{,1} + I_{,2}, \quad I_{,2;2} = -\varepsilon(1 + f)I_{,1} + (f' - \varepsilon I)I_{,2},$$

which implies

$$\begin{aligned} I_2 &= f' I_{,1} + 2I_{,2}, \\ I_{2;2} &= \{f f'' + (f')^2 - 2\varepsilon(1 + f)\}I_{,1} + (3f' - 2\varepsilon I)I_{,2}. \end{aligned}$$

Consequently D^2 of Theorem 11 is written in the form

$$(5.12) \quad D_1 I_{,1} + D_2 I_{,2} = 0,$$

$$(5.12a) \quad \begin{cases} D_1 = \varepsilon\{ff'' + (f')^2\} + 2(f'I - f + 2), \\ D_2 = 3\varepsilon f' + 2I. \end{cases}$$

Proposition 4. *Let F^2 be a two-dimensional Wagner space with non-zero $I_{,2}$. F^2 is a Douglas space, if and only if $I_{,2} = f(I)$ satisfies (5.12).*

We treat only of the sufficient conditions $D_1 = D_2 = 0$ in Proposition 4. $D_2 = 0$ yields $f = c - \varepsilon I^2/3$ with a constant c and then $D_1 = 0$ gives $c = 3/2$. Consequently F^2 has $I_{,2} = 3/2 - \varepsilon I^2/3$, which implies that F^2 is nothing but a Kropina space ([12], [2]).

Up to now we have only two kinds of two-dimensional Wagner spaces whose metrics are known concretely. That is, Kropina metric and cubic metric. A *cubic metric* $L(x, y)$ [18] is defined as

$$L^3(x, y) = a_{ijk}(x)y^i y^j y^k,$$

where a_{ijk} are assumed to be symmetric. A two-dimensional Finsler space F^2 with cubic metric has $I_{,2} = f(I) = -3/2 - 3\varepsilon I^2$ [2], provided that $I_{,2} \neq 0$, which yields $D_1 = 16(3\varepsilon I^2 + 1)$ and $D_2 = -16I$. Thus $D^2 = 0$ is written in the form

$$(5.13) \quad (3\varepsilon I^2 + 1)I_{,1} - II_{,2} = 0.$$

Differentiating (;) this, a procedure similar to (5.11) yields

$$\left(-\frac{31}{2}\varepsilon I - 39I^3\right)I_{,1} + \left(\frac{5}{2} + 13\varepsilon I^2\right)I_{,2} = 0.$$

This together with (5.13) shows that $1 + 2\varepsilon I^2 = 0$, if $I_{,1} = I_{,2} = 0$ do not hold. But $1 + 2\varepsilon I^2 = 0$ leads to $I_{,2} = 0$, a contradiction, and hence we obtain $I_{,1} = I_{,2} = 0$. Then one of the Ricci identities [4]: $S_{,1,2} - S_{,2,1} = -RS_{,2}$, gives rise to $R = 0$, that is, F^2 is a locally Minkowski space, because R is the curvature.

On the other hand, if $I_{,2} = 0$, then Theorem 12 shows that F^2 is a Berwald space with constant main scalar I [18]. Therefore we have the

Theorem 14. *Let F^2 be a two-dimensional Douglas space with cubic metric. Then F^2 is (1) a locally Minkowski space, or (2) a Berwald space with $\varepsilon = -1$, $I^2 = 1/2$ and $L^3 = \beta\gamma^2$, where β and γ are 1-forms.*

6. Douglas spaces with special metric

First we consider the Randers spaces. They, with the Kropina spaces which appeared in Example 1 etc., have played a central role in the theory of (α, β) -metrics, where $\alpha^2 = a_{ij}(x)y^i y^j$ and $\beta = b_i(x)y^i$ (cf. [1], 1.3 and 1.4; [13]; [19]).

For a Finsler space $F^n = (M^n, L(\alpha, \beta))$ with (α, β) -metric the Riemannian space $R^n = (M^n, \alpha)$ is said to associate with F^n . In R^n we have the Levi-Civita connection $(\gamma_j^i{}_k(x))$, in which we have the symbols as follows:

$$\begin{aligned} r_{ij} &= \frac{1}{2}(b_{i;j} + b_{j;i}), & s_{ij} &= \frac{1}{2}(b_{i;j} - b_{j;i}) = \frac{1}{2}(\partial_j b_i - \partial_i b_j), \\ s_j^i &= a^{ir} s_{rj}, & s_i &= b_r s_i^r. \end{aligned}$$

Now we consider a Randers space $F^n = (M^n, L = \alpha + \beta)$. On account of the simplicity of its metric, the functions G^i of F^n are easily written [15] as

$$2G^i = \gamma_0^i{}_0 + 2(Ay^i + \alpha s_0^i),$$

where $A = (r_{00} - 2\alpha s_0)/2(\alpha + \beta)$. Then we get

$$2D^{ij} = (\gamma_0^i{}_0 y^j - \gamma_0^j{}_0 y^i) + 2\alpha(s_0^i y^j - s_0^j y^i).$$

It is obvious that the terms in the first (..) of the right-hand side are homogeneous polynomials in (y^i) of degree three, while the Riemannian α is surely irrational in (y^i) . Therefore F^n is a Douglas space, if and only if $s_0^i y^j - s_0^j y^i = 0$; transvection by $Y_j = a_{jr} y^r$ gives $s_0^i = 0$, that is, $s_{ij} = 0$. Therefore

Theorem 15. *A Randers space is of Douglas type, if and only if $\partial_j b_i - \partial_i b_j = 0$, that is, β is a closed form. Then*

$$2G^i = \gamma_0^i{}_0 + \frac{r_{00}}{\alpha + \beta} y^i,$$

where r_{ij} is equal to $b_{i;j}$.

Remark. It is interesting for the authors to compare Theorem 15 with KIKUCHI's theorem ([10], [15]). A Randers space is a Berwald space if and only if $b_{i;j} = 0$, and then $2G^i = \gamma_0^i$.

We shall pay attention to another special metric, called the *1-form metric* ([1], 1.5). A Finsler metric L is called a 1-form metric, if we have a typical Minkowski metric $L(v^\alpha)$ on an n -dimensional vector space and $L = L(a^\alpha)$, $a^\alpha = a_i^\alpha(x)y^i$ being n 1-forms. Of course, a^α , $\alpha = 1, \dots, n$, should be independent: $d = \det(a_i^\alpha) \neq 0$. Putting $L_\alpha = \partial L / \partial a^\alpha$, we have

$$L_{(i)} = L_\alpha a_i^\alpha, \quad L_{(i)(j)} = L_{\alpha\beta} a_i^\alpha a_j^\beta.$$

We shall restrict our consideration to two-dimensional spaces F^2 with 1-form metric $L(a^1, a^2)$. Analogously to the Weierstrass invariant W of (5.4), because of the homogeneity of $L(a^1, a^2)$ we can also define

$$(6.1) \quad w = \frac{L_{11}}{(a^2)^2} = \frac{-L_{12}}{a^1 a^2} = \frac{L_{22}}{(a^1)^2},$$

called the *intrinsic Weierstrass invariant*. It is easy to show

$$(6.2) \quad W = w d^2.$$

In a general F^n with 1-form metric we have a linear non-symmetric connection $(\Gamma_j^i(x))$, which is called the 1-form connection ([1], 1.5.2) and is given by

$$\Gamma_j^i{}^k = b_\alpha^i \partial_k a_j^\alpha, \quad (b_\alpha^i) = (a_i^\alpha)^{-1}.$$

The definition is rewritten as $a_{i;j}^\alpha = \partial_j a_i^\alpha - a_r^\alpha \Gamma_i^r{}_j = 0$, and hence, in the induced Finsler connection $(\Gamma_j^i{}^k, \Gamma_0^i{}^j)$, we have

$$L_{(j);i} = (L_\alpha a_j^\alpha)_{;i} = 0 = L_{(j)i} - L_{(j)(r)} \Gamma_0^r{}_i - L_{(r)} \Gamma_j^r{}_i.$$

Now, let us return to F^2 with 1-form metric, and we have

$$\begin{aligned} M &= L_{1(2)} - L_{2(1)} = L_{(2)(r)} \Gamma_0^r{}_1 - L_{(1)(r)} \Gamma_0^r{}_2 - L_{(r)} T_1^{(r)}{}_2 \\ &= W(y^1 \Gamma_0^2{}_0 - y^2 \Gamma_0^1{}_0) - L_\alpha T^\alpha, \quad \alpha = 1, 2, \end{aligned}$$

where $T_i^r{}_j = b_\alpha^r (\partial_j a_i^\alpha - \partial_i a_j^\alpha)$ is the torsion tensor of the 1-form connection and $T^\alpha = a_r^\alpha T_1^r{}_2 = \partial_2 a_1^\alpha - \partial_1 a_2^\alpha$. It is noted that the terms in the parentheses are homogeneous polynomials in (y^1, y^2) of degree three.

Therefore Theorem 10 together with (6.2) shows the

Theorem 16. *Let F^2 be a two-dimensional Finsler space with 1-form metric $L(a^1, a^2)$. F^2 is a Douglas space, if and only if $L_\alpha T^\alpha/w$ is a homogeneous polynomial in (y^1, y^2) of degree three, where*

$$L_\alpha = \partial L / \partial a^\alpha, \quad T^\alpha = \partial_2 a_1^\alpha - \partial_1 a_2^\alpha, \quad \alpha = 1, 2,$$

and w is the intrinsic Weierstrass invariant.

It is noted from (5.7) that for an F^2 with 1-form metric the Weierstrass form of geodesics equation is written in the form

$$(6.3) \quad p\dot{q} - \dot{p}q + p\Gamma_0^2{}_0 - q\Gamma_0^1{}_0 - \frac{1}{wd^2}L_\alpha T^\alpha = 0, \quad \alpha = 1, 2.$$

Example 4. As is well-known ([5]; [13], §28; [1], 3.5), any two-dimensional Berwald spaces with constant main scalar are with 1-form metric. Except those trivial spaces, let us here treat of an F^2 with the metric

$$L = a^2 \log \left| \frac{a^2}{a^1} \right|^r,$$

where r is a non-zero real number. We have

$$\begin{aligned} L_1 &= -r \frac{a^2}{a^1}, \quad L_2 = r \left(1 + \log \left| \frac{a^2}{a^1} \right| \right), \quad L_{12} = -\frac{r}{a^1}, \\ w &= \frac{r}{(a^1)^2 a^2}, \quad \frac{1}{w} L_\alpha T^\alpha = -a^1 (a^2)^2 T^1 + (a^1)^2 a^2 \left(1 + \log \left| \frac{a^2}{a^1} \right| \right) T^2. \end{aligned}$$

Since T^α do not contain (p, q) , $L_\alpha T^\alpha/w$ is homogeneous polynomial in (p, q) of degree three, if and only if $T^2 = 0$; $(a_1^2)_y - (a_2^2)_x = 0$. Therefore this F^2 is a Douglas space, if and only if the form a^2 is closed, and then the equation of the geodesics is written from (6.3) as

$$p\dot{q} - \dot{p}q + p\Gamma_0^2{}_0 - q\Gamma_0^1{}_0 + a^1 (a^2)^2 \frac{T^1}{(d)^2} = 0,$$

where $T^1 = (a_1^1)_y - (a_2^1)_x$.

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