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A common fixed point theorem for four mappings satisfying a rational inequality

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Abstract. A metrical common fixed point theorem for four mappings satisfying a rational inequality has been obtained which generalizes and unifies several previously known results. Two illustrative examples are also furnished.

SESSA [4] defined the pair of self mappings $\{S, I\}$ on a metric space (X, d) to be weakly commuting if $d(SIx, ISx) \leq d(Ix, Sx)$ for all x in X. It is obvious that two commuting mappings are weakly commuting but two weakly commuting mappings do not necessarily commute as shown in Example 1 of [4].

We prove the following theorem which unifies the results of FISHER [2] and FISHER [3]. In the process a result of AQEEL and SHAKIL [1] is generalized and improved.

Theorem 1. *Let $\{S, I\}$ and $\{T, J\}$ be two weakly commuting pairs of mappings of a complete metric space (X, d) into itself such that*

$$T(X) \subset I(X),\ S(X) \subset J(X); \tag{1}$$

and for all x, y in X either

$$d(Sx, Ty) \leq \frac{\alpha\left[\{d(Sx, Ix)\}^2 + \{d(Ty, Jy)\}^2\right]}{d(Sx, Ix) + d(Ty, Jy)} + \beta d(Ix, Jy) \tag{2}$$

if $d(Sx, Ix) + d(Ty, Jy) \neq 0$ where $\alpha, \beta > 0$, $\alpha + \beta < 1$,
or

$$d(Sx, Ty) = 0 \quad \text{if} \quad d(Sx, Ix) + d(Ty, Jy) = 0. \tag{2'}$$

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If one of S, T, I or J is continuous, then S, T, I and J have a unique common fixed point z. Further, z is the unique common fixed point of S and I and of T and J.

PROOF. Let x_0 be an arbitrary point in X. As $S(X)$ is contained in $J(X)$, we can choose a point x_1 in X such that $Sx_0 = Jx_1$. Since $T(X)$ is also contained in $I(X)$, we can choose a point x_2 in X such that $Tx_1 = Ix_2$. In this way, we can choose x_{2n}, x_{2n+1}, x_{2n+2} such that $Sx_{2n} = Jx_{2n+1}$ and $Tx_{2n+1} = Ix_{2n+2}$ for $n = 0, 1, 2, \ldots$
Let us denote $U_{2n} = d(Sx_{2n}, Tx_{2n+1})$ and $U_{2n+1} = d(Tx_{2n+1}, Sx_{2n+2})$. We distinguish two cases:

(i) Suppose $U_{2n} + U_{2n+1} \neq 0$ for $n = 0, 1, 2, \ldots$; then on using inequality (2), we get

$$U_{2n+1} \leq \frac{\alpha\{(U_{2n})^2 + (U_{2n+1})^2\}}{U_{2n} + U_{2n+1}} + \beta U_{2n}, \tag{3}$$

so that

$$(1-\alpha)U_{2n+1}^2 + (1-\beta)U_{2n}U_{2n+1} - (\alpha+\beta)U_{2n}^2 \leq 0.$$

The positive root K of the quadratic equation $(1-\alpha)t^2 + (1-\beta)t - (\alpha+\beta) = 0$ is

$$\left[\{(1-\beta)^2 + 4(\alpha+\beta)(1-\alpha)\}^{1/2} - (1-\beta)\right] / (2-2\alpha)$$

and since $\alpha + \beta < 1$, it follows that $K < 1$. Thus $U_{2n+1} \leq KU_{2n}$.

Similarly if $U_{2n} + U_{2n-1} \neq 0$, $n = 1, 2, 3, \ldots$, then the inequality

$$U_{2n} \leq \frac{\alpha\left\{(U_{2n-1})^2 + (U_{2n})^2\right\}}{U_{2n-1} + U_{2n}} + \beta U_{2n-1},$$

like the earlier one, gives

$$U_{2n} \leq KU_{2n-1}.$$

Thus, in general, we have shown that for $k = 0, 1, 2, \ldots,$

$$U_{k+1} \leq \frac{\alpha\left\{(U_k)^2 + (U_{k+1})^2\right\}}{U_k + U_{k+1}} + \beta U_k.$$

Having this we see that $U_{k+1} \leq KU_k$ which yields $U_k \leq K^k U_0$. Now it follows that the sequence

$$\{Sx_0, Tx_1, Sx_2, \ldots, Tx_{2n-1},\ Sx_{2n},\ Tx_{2n+1}, \ldots\} \tag{4}$$

is a Cauchy sequence in the complete metric space (X, d) and so has a limit point z in X. Hence the sequences

$$\{Sx_{2n}\} = \{Jx_{2n+1}\} \quad \text{and} \quad \{Tx_{2n-1}\} = \{Ix_{2n}\}$$

which are subsequences of (4) also converge to the point z.

Let us suppose that I is continuous so that the sequences $\{I^2x_{2n}\}$ and $\{ISx_{2n}\}$ converge to the point Iz. Since S and I are weakly commuting, we have

$$d(SIx_{2n},\ ISx_{2n}) \leq d(Ix_{2n},\ Sx_{2n})$$

and so the sequence $\{SIx_{2n}\}$ also converges to the point Iz.

We now have

$$d(SIx_{2n},\ Tx_{2n+1}) \leq \frac{\alpha\left[\{d(I^2x_{2n}, SIx_{2n})\}^2 + \{d(Tx_{2n+1}, Jx_{2n+1})\}^2\right]}{d(I^2x_{2n}, SIx_{2n}) + d(Tx_{2n+1}, Jx_{2n+1})} + \\ + \beta d(I^2x_{2n}, Jx_{2n+1}).$$

Letting $n \to \infty$, we have

$$d(Iz, z) \leq \beta d(Iz, z),$$

a contradiction. It follows that $Iz = z$. Further

$$d(Sz, Tx_{2n+1}) \leq \frac{\alpha\left[\{d(Iz, Sz)\}^2 + \{d(Tx_{2n+1}, Jx_{2n+1})\}^2\right]}{d(Iz, Sz) + d(Tx_{2n+1}, Jx_{2n+1})} + \\ + \beta d(Iz, Jx_{2n+1})$$

and letting $n \to \infty$, we get

$$d(Sz, z) \leq \alpha d(Sz, z),$$

again a contradiction. Hence $Sz = z$.

This means that z is in the range of S and since the range of J contains the range of S, there exists a point z' such that $Jz' = z$. Thus

$$d(z, Tz') = d(Sz, Tz') \leq$$

$$\leq \frac{\alpha\left[\{d(Sz, Iz)\}^2 + \{d(Tz', Jz')\}^2\right]}{d(Sz, Iz) + d(Tz', Jz')} + \beta d(Iz, Jz') =$$

$$= \alpha d(z, Tz') < d(z, Tz'),$$

which implies that $Tz' = z$.

Since T and J weakly commute,

$$d(Tz, Jz) = d(TJz', JTz') \leq d(Jz', Tz') = d(z, z) = 0$$

giving thereby $Tz = Jz$ and so

$$d(z, Tz) = d(Sz, Tz) \leq$$

$$\leq \frac{\alpha\left[\{d(Iz, Sz)\}^2 + \{d(Tz, Jz)\}^2\right]}{d(Iz, Sz) + d(Tz, Jz)} + \beta d(Iz, Jz) = 0$$

which implies that $z = Tz = Jz$.

We therefore have proved that z is a common fixed point of S, T, I and J.

Now suppose that S is continuous, so that the sequences $\{S^2x_{2n}\}$ and $\{SIx_{2n}\}$ converge to Sz. Since S and I weakly commute, it follows as above that the sequence $\{ISx_{2n}\}$ also converges to Sz. Thus

$$d(S^2x_{2n}, Tx_{2n+1}) \leq \frac{\alpha\left[\{d(S^2x_{2n}, ISx_{2n})\}^2 + \{d(Tx_{2n+1}, Jx_{2n+1})\}^2\right]}{d(S^2x_{2n}, ISx_{2n}) + d(Tx_{2n+1}, Jx_{2n+1})} + \\ + \beta d(ISx_{2n}, Jx_{2n+1}).$$

Letting $n \to \infty$, we have

$$d(Sz, z) \leq \beta d(Sz, z) < d(Sz, z).$$

It follows that $Sz = z$.

Once again, there exists a point z' in X such that $Jz' = z$.

Thus

$$d(S^2x_{2n}, Tz') \leq \frac{\alpha\left[\{d(S^2x_{2n}, ISx_{2n})\}^2 + \{d(Tz', Jz')\}^2\right]}{d(S^2x_{2n}, ISx_{2n}) + d(Tz', Jz')} + \\ + \beta d(ISx_{2n}, Jz').$$

Letting $n \to \infty$, we have

$$d(z, Tz') \leq \alpha d(z, Tz'),$$

so that $z = Tz'$.

Since T and J weakly commute, it again follows as above that $Tz = Jz$.

Further,

$$d(Sx_{2n}, Tz) \leq \frac{\alpha\left[\{d(Sx_{2n}, Ix_{2n})\}^2 + \{d(Tz, Jz)\}^2\right]}{d(Sx_{2n}, Ix_{2n}) + d(Tz, Jz)} + \beta d(Ix_{2n}, Jz).$$

Letting $n \to \infty$, we have

$$d(z, Tz) \leq \beta d(z, Tz)$$

and so $z = Tz = Jz$.

The point z therefore is in the range of T and since the range of I contains the range of T, there exists a point z'' in X such that $Iz'' = z$. Thus

$$d(Sz'', z) = d(Sz'', Tz) \leq \frac{\alpha\left[\{d(Sz'', Iz'')\}^2 + \{d(Tz, Jz)\}^2\right]}{d(Sz'', Iz'') + d(Tz, Jz)} + \\ + \beta d(Iz'', Jz) = \alpha d(Sz'', z)$$

and so $Sz'' = z$.

Again, since S and I weakly commute, we have

$$d(Sz, Iz) = d(SIz'', ISz'') \leq d(Iz'', Sz'') = d(z, z) = 0.$$

Thus $Sz = Iz = z$.

We thus have proved again that z is a common fixed point of S, T, I and J.

If the mapping T or J is continuous instead of S or I then the proof that z is a common fixed point of S, T, I and J is similar.

(ii) Suppose $U_{2n} + U_{2n+1} = 0$. Then, for some n, the inequality (3) gives

$U_{2n} = d(Sx_{2n},\ Tx_{2n+1}) = 0$, and
$U_{2n+1} = d(Tx_{2n+1},\ Sx_{2n+2}) = 0$, giving thereby
$Sx_{2n} = Jx_{2n+1} = Tx_{2n+1} = Sx_{2n+2} = \cdots = z$.

Now we assert that there exists a point w such that $Sw = Iw = Tw = Jw = z$ because if $Sw = Iw \neq z$, then

$$\begin{aligned} 0 < d(Iw, z) = d(Sw, Tx_{2n+1}) \leq \\ \leq \frac{\alpha\left[\{d(Sw, Iw)\}^2 + \{d(Tx_{2n+1}, Jx_{2n+1})\}^2\right]}{d(Sw, Iw) + d(Tx_{2n+1}, Jx_{2n+1})} + \\ + \beta(Iw, Jx_{2n+1}) < d(Iw, z) \end{aligned}$$

which yields that $Iw = z = Sw$. Similarly, one can argue that $Tw = Jw = z$.

Now suppose that I or S is continuous. Proceeding as above, it can be shown that $Iw = z$ is a common fixed point of S, T, I and J.

Furthermore, if J or T is continuous, then the proof that z is a common fixed point of S, T, I and J is similar.

In order to prove the uniqueness of the common fixed point z, let w be a second common fixed point of S and I. Then

$$\begin{aligned} d(w, z) = d(Sw, Tz) \leq \\ \leq \frac{\alpha\left[\{d(Sw, Iw)\}^2 + \{d(Tz, Jz)\}^2\right]}{d(Sw, Iw) + d(Tz, Jz)} + \beta d(Iw, Jz) = 0, \end{aligned}$$

which yields that $w = z$.

Similarly it can be proved that z is a unique common fixed point of T and J. This completes the proof.

Finally we furnish examples in order to demonstrate the validity of the hypotheses and to show the degree of generality of our theorem.

Example 1. Let $X = \{1,2,3,4\}$ be a finite set with a metric d given by

$$d(1,1) = d(2,2) = d(3,3) = d(4,4) = 0,$$
$$d(1,2) = d(1,4) = d(2,3) = 5,$$
$$d(1,3) = 4,\ d(2,4) = 6,\ d(3,4) = 7.$$

Define I, J, S and T on X by

$$I1 = 1,\ I2 = 3,\ I3 = I4 = 4$$
$$J1 = 1,\ J2 = J3 = 2,\ J4 = 3$$
$$S1 = S2 = S3 = 1,\ S4 = 2$$
$$T1 = T2 = T3 = T4 = 1.$$

Since $SI1 = 1 = IS1$, $SI2 = 1 = IS2$, $SI3 = 2 \neq 1 = IS3$ and $SI4 = 2 \neq 3 = IS4$, the pair $\{S, I\}$ is not commuting but it is weakly commuting because $5 = d(SI3, IS3) \leq d(I3, S3) = 5$ and $5 = d(SI4, IS4) \leq d(I4, S4) = 6$.

Also the pair $\{T, J\}$ is commuting and hence weakly commuting, clearly I (or S) is continuous and
$S(X) = \{1,2\} \subset \{1,2,3\} = J(X)$ and
$T(X) = \{1\} \subset \{1,4\} = I(X)$.

A routine calculation shows that the inequalities (2) and (2') are satisfied for all $x, y \in X$. Hence all the conditions of Theorem 1 are satisfied and 1 is the unique common fixed point of S, I, T and J. Also 1 is the unique common fixed point of S and I and of T and J. Here it is interesting to note that the mappings I and J have two fixed points each.

However, Theorem 1 seems to be a genuine extension to the theorem of Fisher [3], because if we take $x = 4$ and $y = 1$, then condition $d(Sx, Sy) \leq kd(Ix, Jy)$ implies that $5 \leq k \cdot 5$, which is a contradiction as $0 \leq k < 1$.

Example 2. Consider $X = [0,1]$ with the usual metric. Define the self-mappings S, T, I and J on X as $Sx = \frac{x}{4}$, $Tx = \frac{x}{5}$, $Ix = \frac{x}{2}$ and $Jx = \frac{3}{4}x$. Clearly $T(X) = \left[0, \frac{1}{5}\right] \subset I(X) = \left[0, \frac{1}{2}\right]$ and $S(X) = \left[0, \frac{1}{4}\right] \subset J(X) = \left[0, \frac{3}{4}\right]$. Also the pairs of mappings $\{S, I\}$ and $\{T, J\}$ are commuting hence weakly commuting.

It is a routine to verify that the condition (2) of Theorem 1 is satisfied for all x, y in X with $\alpha = \frac{1}{20}$ and $\beta = \frac{1}{5}$. Clearly 0 is the unique common fixed point of S, T, I and J.

However, our unification is genuine because for $x = 1$ and $y = 0$, the condition (2) with $\beta = 0$ implies $\frac{1}{4} \leq \alpha\frac{1}{4}$ whereas condition (2) with $\alpha = 0$ implies $\frac{1}{4} \leq \beta\frac{1}{8}$ which are not possible with the above mentioned values of α and β.

Remark 1. In Theorem 1, if we set $\beta = 0$ and $I = J =$ identity mapping, then we get Theorem 2 of FISHER [2].

Remark 2. If $I = J =$ identity mapping, then Theorem 1 reduces to Theorem 2.1 of AQEEL — SHAKIL [1].

Remark 3. By setting $\alpha = 0$ in our Theorem 1 we get an improved version of the theorem of FISHER [3]. Note that the theorem of FISHER [3] involves only a triad of mappings.

Remark 4. It may be seen from the proof that if condition (2') of Theorem 1 is omitted then z is the coincidence point of S, T, I and J .

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