

A common fixed point theorem for four mappings satisfying a rational inequality

By AQEEL AHMAD (Aligarh) and M. IMDAD (Aligarh)

Abstract. A metrical common fixed point theorem for four mappings satisfying a rational inequality has been obtained which generalizes and unifies several previously known results. Two illustrative examples are also furnished.

SESSA [4] defined the pair of self mappings $\{S, I\}$ on a metric space (X, d) to be weakly commuting if $d(SIx, ISx) \leq d(Ix, Sx)$ for all x in X . It is obvious that two commuting mappings are weakly commuting but two weakly commuting mappings do not necessarily commute as shown in Example 1 of [4].

We prove the following theorem which unifies the results of FISHER [2] and FISHER [3]. In the process a result of AQEEL and SHAKIL [1] is generalized and improved.

Theorem 1. Let $\{S, I\}$ and $\{T, J\}$ be two weakly commuting pairs of mappings of a complete metric space (X, d) into itself such that

$$(1) \quad T(X) \subset I(X), \quad S(X) \subset J(X);$$

and for all x, y in X either

$$(2) \quad d(Sx, Ty) \leq \frac{\alpha [\{d(Sx, Ix)\}^2 + \{d(Ty, Jy)\}^2]}{d(Sx, Ix) + d(Ty, Jy)} + \beta d(Ix, Jy)$$

if $d(Sx, Ix) + d(Ty, Jy) \neq 0$ where $\alpha, \beta > 0$, $\alpha + \beta < 1$,
or

$$(2') \quad d(Sx, Ty) = 0 \quad \text{if} \quad d(Sx, Ix) + d(Ty, Jy) = 0.$$

Key words and Phrases: Common fixed point, coincidence point, Weakly commuting pair,

AMS (MOS) Subject classification (1980): Primary 54H25, Secondary 47H10.

If one of S, T, I or J is continuous, then S, T, I and J have a unique common fixed point z . Further, z is the unique common fixed point of S and I and of T and J .

PROOF. Let x_0 be an arbitrary point in X . As $S(X)$ is contained in $J(X)$, we can choose a point x_1 in X such that $Sx_0 = Jx_1$. Since $T(X)$ is also contained in $I(X)$, we can choose a point x_2 in X such that $Tx_1 = Ix_2$. In this way, we can choose $x_{2n}, x_{2n+1}, x_{2n+2}$ such that $Sx_{2n} = Jx_{2n+1}$ and $Tx_{2n+1} = Ix_{2n+2}$ for $n = 0, 1, 2, \dots$.

Let us denote $U_{2n} = d(Sx_{2n}, Tx_{2n+1})$ and $U_{2n+1} = d(Tx_{2n+1}, Sx_{2n+2})$. We distinguish two cases:

(i) Suppose $U_{2n} + U_{2n+1} \neq 0$ for $n = 0, 1, 2, \dots$; then on using inequality (2), we get

$$(3) \quad U_{2n+1} \leq \frac{\alpha \{(U_{2n})^2 + (U_{2n+1})^2\}}{U_{2n} + U_{2n+1}} + \beta U_{2n},$$

so that

$$(1 - \alpha)U_{2n+1}^2 + (1 - \beta)U_{2n}U_{2n+1} - (\alpha + \beta)U_{2n}^2 \leq 0.$$

The positive root K of the quadratic equation $(1 - \alpha)t^2 + (1 - \beta)t - (\alpha + \beta) = 0$ is

$$\left[\{(1 - \beta)^2 + 4(\alpha + \beta)(1 - \alpha)\}^{1/2} - (1 - \beta) \right] / (2 - 2\alpha)$$

and since $\alpha + \beta < 1$, it follows that $K < 1$. Thus $U_{2n+1} \leq KU_{2n}$.

Similarly if $U_{2n} + U_{2n-1} \neq 0$, $n = 1, 2, 3, \dots$, then the inequality

$$U_{2n} \leq \frac{\alpha \{(U_{2n-1})^2 + (U_{2n})^2\}}{U_{2n-1} + U_{2n}} + \beta U_{2n-1},$$

like the earlier one, gives

$$U_{2n} \leq KU_{2n-1}.$$

Thus, in general, we have shown that for $k = 0, 1, 2, \dots$,

$$U_{k+1} \leq \frac{\alpha \{(U_k)^2 + (U_{k+1})^2\}}{U_k + U_{k+1}} + \beta U_k.$$

Having this we see that $U_{k+1} \leq KU_k$ which yields $U_k \leq K^k U_0$. Now it follows that the sequence

$$(4) \quad \{Sx_0, Tx_1, Sx_2, \dots, Tx_{2n-1}, Sx_{2n}, Tx_{2n+1}, \dots\}$$

is a Cauchy sequence in the complete metric space (X, d) and so has a limit point z in X . Hence the sequences

$$\{Sx_{2n}\} = \{Jx_{2n+1}\} \quad \text{and} \quad \{Tx_{2n-1}\} = \{Ix_{2n}\}$$

which are subsequences of (4) also converge to the point z .

Let us suppose that I is continuous so that the sequences $\{I^2x_{2n}\}$ and $\{ISx_{2n}\}$ converge to the point Iz . Since S and I are weakly commuting, we have

$$d(SIx_{2n}, ISx_{2n}) \leq d(Ix_{2n}, Sx_{2n})$$

and so the sequence $\{SIx_{2n}\}$ also converges to the point Iz .

We now have

$$d(SIx_{2n}, Tx_{2n+1}) \leq \frac{\alpha \left[\{d(I^2x_{2n}, SIx_{2n})\}^2 + \{d(Tx_{2n+1}, Jx_{2n+1})\}^2 \right]}{d(I^2x_{2n}, SIx_{2n}) + d(Tx_{2n+1}, Jx_{2n+1})} + \beta d(I^2x_{2n}, Jx_{2n+1}).$$

Letting $n \rightarrow \infty$, we have

$$d(Iz, z) \leq \beta d(Iz, z),$$

a contradiction. It follows that $Iz = z$. Further

$$d(Sz, Tx_{2n+1}) \leq \frac{\alpha \left[\{d(Iz, Sz)\}^2 + \{d(Tx_{2n+1}, Jx_{2n+1})\}^2 \right]}{d(Iz, Sz) + d(Tx_{2n+1}, Jx_{2n+1})} + \beta d(Iz, Jx_{2n+1})$$

and letting $n \rightarrow \infty$, we get

$$d(Sz, z) \leq \alpha d(Sz, z),$$

again a contradiction. Hence $Sz = z$.

This means that z is in the range of S and since the range of J contains the range of S , there exists a point z' such that $Jz' = z$. Thus

$$\begin{aligned} d(z, Tz') &= d(Sz, Tz') \leq \\ &\leq \frac{\alpha \left[\{d(Sz, Iz)\}^2 + \{d(Tz', Jz')\}^2 \right]}{d(Sz, Iz) + d(Tz', Jz')} + \beta d(Iz, Jz') = \\ &= \alpha d(z, Tz') < d(z, Tz'), \end{aligned}$$

which implies that $Tz' = z$.

Since T and J weakly commute,

$$d(Tz, Jz) = d(TJz', JTz') \leq d(Jz', Tz') = d(z, z) = 0$$

giving thereby $Tz = Jz$ and so

$$\begin{aligned} d(z, Tz) &= d(Sz, Tz) \leq \\ &\leq \frac{\alpha \left[\{d(Iz, Sz)\}^2 + \{d(Tz, Jz)\}^2 \right]}{d(Iz, Sz) + d(Tz, Jz)} + \beta d(Iz, Jz) = 0 \end{aligned}$$

which implies that $z = Tz = Jz$.

We therefore have proved that z is a common fixed point of S , T , I and J .

Now suppose that S is continuous, so that the sequences $\{S^2x_{2n}\}$ and $\{SIx_{2n}\}$ converge to Sz . Since S and I weakly commute, it follows as above that the sequence $\{ISx_{2n}\}$ also converges to Sz . Thus

$$d(S^2x_{2n}, Tx_{2n+1}) \leq \frac{\alpha \left[\{d(S^2x_{2n}, ISx_{2n})\}^2 + \{d(Tx_{2n+1}, Jx_{2n+1})\}^2 \right]}{d(S^2x_{2n}, ISx_{2n}) + d(Tx_{2n+1}, Jx_{2n+1})} + \\ + \beta d(ISx_{2n}, Jx_{2n+1}).$$

Letting $n \rightarrow \infty$, we have

$$d(Sz, z) \leq \beta d(Sz, z) < d(Sz, z).$$

It follows that $Sz = z$.

Once again, there exists a point z' in X such that $Jz' = z$.

Thus

$$d(S^2x_{2n}, Tz') \leq \frac{\alpha \left[\{d(S^2x_{2n}, ISx_{2n})\}^2 + \{d(Tz', Jz')\}^2 \right]}{d(S^2x_{2n}, ISx_{2n}) + d(Tz', Jz')} + \\ + \beta d(ISx_{2n}, Jz').$$

Letting $n \rightarrow \infty$, we have

$$d(z, Tz') \leq \alpha d(z, Tz'),$$

so that $z = Tz'$.

Since T and J weakly commute, it again follows as above that $Tz = Jz$.

Further,

$$d(Sx_{2n}, Tz) \leq \frac{\alpha \left[\{d(Sx_{2n}, Ix_{2n})\}^2 + \{d(Tz, Jz)\}^2 \right]}{d(Sx_{2n}, Ix_{2n}) + d(Tz, Jz)} + \beta d(Ix_{2n}, Jz).$$

Letting $n \rightarrow \infty$, we have

$$d(z, Tz) \leq \beta d(z, Tz)$$

and so $z = Tz = Jz$.

The point z therefore is in the range of T and since the range of I contains the range of T , there exists a point z'' in X such that $Iz'' = z$. Thus

$$d(Sz'', z) = d(Sz'', Tz) \leq \frac{\alpha \left[\{d(Sz'', Iz'')\}^2 + \{d(Tz, Jz)\}^2 \right]}{d(Sz'', Iz'') + d(Tz, Jz)} + \\ + \beta d(Iz'', Jz) = \alpha d(Sz'', z)$$

and so $Sz'' = z$.

Again, since S and I weakly commute, we have

$$d(Sz, Iz) = d(SIz'', ISz'') \leq d(Iz'', Sz'') = d(z, z) = 0.$$

Thus $Sz = Iz = z$.

We thus have proved again that z is a common fixed point of S , T , I and J .

If the mapping T or J is continuous instead of S or I then the proof that z is a common fixed point of S , T , I and J is similar.

(ii) Suppose $U_{2n} + U_{2n+1} = 0$. Then, for some n , the inequality (3) gives

$$U_{2n} = d(Sx_{2n}, Tx_{2n+1}) = 0, \text{ and}$$

$$U_{2n+1} = d(Tx_{2n+1}, Sx_{2n+2}) = 0, \text{ giving thereby}$$

$$Sx_{2n} = Jx_{2n+1} = Tx_{2n+1} = Sx_{2n+2} = \cdots = z.$$

Now we assert that there exists a point w such that $Sw = Iw = Tw = Jw = z$ because if $Sw = Iw \neq z$, then

$$\begin{aligned} 0 < d(Iw, z) &= d(Sw, Tx_{2n+1}) \leq \\ &\leq \frac{\alpha \left[\{d(Sw, Iw)\}^2 + \{d(Tx_{2n+1}, Jx_{2n+1})\}^2 \right]}{d(Sw, Iw) + d(Tx_{2n+1}, Jx_{2n+1})} + \\ &\quad + \beta(Iw, Jx_{2n+1}) < d(Iw, z) \end{aligned}$$

which yields that $Iw = z = Sw$. Similarly, one can argue that $Tw = Jw = z$.

Now suppose that I or S is continuous. Proceeding as above, it can be shown that $Iw = z$ is a common fixed point of S , T , I and J .

Furthermore, if J or T is continuous, then the proof that z is a common fixed point of S , T , I and J is similar.

In order to prove the uniqueness of the common fixed point z , let w be a second common fixed point of S and I . Then

$$\begin{aligned} d(w, z) &= d(Sw, Tz) \leq \\ &\leq \frac{\alpha \left[\{d(Sw, Iw)\}^2 + \{d(Tz, Jz)\}^2 \right]}{d(Sw, Iw) + d(Tz, Jz)} + \beta d(Iw, Jz) = 0, \end{aligned}$$

which yields that $w = z$.

Similarly it can be proved that z is a unique common fixed point of T and J . This completes the proof.

Finally we furnish examples in order to demonstrate the validity of the hypotheses and to show the degree of generality of our theorem.

Example 1. Let $X = \{1, 2, 3, 4\}$ be a finite set with a metric d given by

$$d(1, 1) = d(2, 2) = d(3, 3) = d(4, 4) = 0,$$

$$d(1, 2) = d(1, 4) = d(2, 3) = 5,$$

$$d(1, 3) = 4, \quad d(2, 4) = 6, \quad d(3, 4) = 7.$$

Define I, J, S and T on X by

$$I1 = 1, \quad I2 = 3, \quad I3 = I4 = 4$$

$$J1 = 1, \quad J2 = J3 = 2, \quad J4 = 3$$

$$S1 = S2 = S3 = 1, \quad S4 = 2$$

$$T1 = T2 = T3 = T4 = 1.$$

Since $SI1 = 1 = IS1$, $SI2 = 1 = IS2$, $SI3 = 2 \neq 1 = IS3$ and $SI4 = 2 \neq 3 = IS4$, the pair $\{S, I\}$ is not commuting but it is weakly commuting because $5 = d(SI3, IS3) \leq d(I3, S3) = 5$ and $5 = d(SI4, IS4) \leq d(I4, S4) = 6$.

Also the pair $\{T, J\}$ is commuting and hence weakly commuting, clearly I (or S) is continuous and

$$S(X) = \{1, 2\} \subset \{1, 2, 3\} = J(X) \text{ and}$$

$$T(X) = \{1\} \subset \{1, 4\} = I(X).$$

A routine calculation shows that the inequalities (2) and (2') are satisfied for all $x, y \in X$. Hence all the conditions of Theorem 1 are satisfied and 1 is the unique common fixed point of S, I, T and J . Also 1 is the unique common fixed point of S and I and of T and J . Here it is interesting to note that the mappings I and J have two fixed points each.

However, Theorem 1 seems to be a genuine extension to the theorem of Fisher [3], because if we take $x = 4$ and $y = 1$, then condition $d(Sx, Sy) \leq kd(Ix, Jy)$ implies that $5 \leq k \cdot 5$, which is a contradiction as $0 \leq k < 1$.

Example 2. Consider $X = [0, 1]$ with the usual metric. Define the self-mappings S, T, I and J on X as $Sx = \frac{x}{4}$, $Tx = \frac{x}{5}$, $Ix = \frac{x}{2}$ and $Jx = \frac{3}{4}x$. Clearly $T(X) = [0, \frac{1}{5}] \subset I(X) = [0, \frac{1}{2}]$ and $S(X) = [0, \frac{1}{4}] \subset J(X) = [0, \frac{3}{4}]$. Also the pairs of mappings $\{S, I\}$ and $\{T, J\}$ are commuting hence weakly commuting.

It is a routine to verify that the condition (2) of Theorem 1 is satisfied for all x, y in X with $\alpha = \frac{1}{20}$ and $\beta = \frac{1}{5}$. Clearly 0 is the unique common fixed point of S, T, I and J .

However, our unification is genuine because for $x = 1$ and $y = 0$, the condition (2) with $\beta = 0$ implies $\frac{1}{4} \leq \alpha \frac{1}{4}$ whereas condition (2) with $\alpha = 0$ implies $\frac{1}{4} \leq \beta \frac{1}{8}$ which are not possible with the above mentioned values of α and β .

Remark 1. In Theorem 1, if we set $\beta = 0$ and $I = J =$ identity mapping, then we get Theorem 2 of FISHER [2].

Remark 2. If $I = J =$ identity mapping, then Theorem 1 reduces to Theorem 2.1 of AQEEL — SHAKIL [1].

Remark 3. By setting $\alpha = 0$ in our Theorem 1 we get an improved version of the theorem of FISHER [3]. Note that the theorem of FISHER [3] involves only a triad of mappings.

Remark 4. It may be seen from the proof that if condition (2') of Theorem 1 is omitted then z is the coincidence point of S, T, I and J .

Acknowledgement. The authors are grateful to the learned referee for his helpful comments regarding the presentation of the paper.

References

- [1] A. AHMAD and M. SHAKIL, Some fixed point theorems satisfying a rational Inequality, (in appear).
- [2] B. FISHER, Common fixed point and constant mappings satisfying a rational inequality, *Math. Sem. Notes* 6 (1978), 29–35.
- [3] B. FISHER, Mapping with a common fixed points, *Math. Sem. Notes* 7 (1979), 81–84.
- [4] S. SESSA, On a weak commutativity condition of mappings in fixed point considerations, *Publ. Inst. Math.* 32 (46) (1982), 149–153.

AQEEL AHMAD
DEPARTMENT OF APPLIED MATHEMATICS
Z.H. COLLEGE OF ENGG. AND TECH.,
A.M.U., ALIGARH-202002, INDIA.

M. IMDAD
DEPARTMENT OF MATHEMATICS
ALIGARH MUSLIM UNIVERSITY
ALIGARH-202002, INDIA.

(Received April 23, 1990; revised February 25, 1991)