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On a Theorem of Bovdi

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1. Introduction

The purpose of this note is to generalize and reinterpret a result of BOVDI [3] concerning units in a commutative group ring $\mathbf{Z}A$ which are "unitary" with respect to a quadratic character $A \to \{\pm 1\}$ of the underlying (finite) abelian group A. It will be apparent from our account that one need not restrict one's attention to cyclic A, as was done in [3].

Given an arbitrary character $\psi : A \to K^\times$, where K is a suitable cyclotomic field, let $\tilde{\psi}$ denote the obvious "twist-by-ψ" automorphism on the group algebra KA, which is defined by

$$\tilde{\psi}\left(\sum_{x\in A} a_x \cdot x\right) = \sum_{x\in A} a_x \psi(x) \cdot x \ . \tag{1}$$

If the character ψ has order q, then so does the automorphism $\tilde{\psi}$. We are particularly interested in the *norm* N_ψ with respect to this automorphism:

$$N_\psi(u) = \prod_{i=0}^{q-1} \tilde{\psi}^i(u) \ , \tag{2}$$

applied to elements $u \in \mathbf{Z}A$. Since the coefficients of $N_\psi(u)$ are algebraic integers fixed under the action of $\mathrm{Gal}(K/\mathbf{Q})$, we have $N_\psi(u)$ again in $\mathbf{Z}A$. In particular, N_ψ may be regarded as an endomorphism of the unit group $U\mathbf{Z}A$.

An element $u \in \mathbf{Z}A$ will be called ψ-normal, if $u \in \ker N_\psi$, i.e. if $N_\psi(u) = 1$. For quadratic ψ, this property is closely related to that of being ψ-unitary in the sense of [3]. In Section 2 below, we shall explicitly construct a free abelian subgroup of finite index in $\ker N_\psi$, and thence recover Bovdi's theorem about ψ-unitary units in the quadratic case.

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We conclude this introduction by recalling the construction, due to BASS and MILNOR, of nice subgroups of finite index in $U_1\mathbf{Z}A$, the group of units with coefficient sum 1.

Starting with a cyclic group C having n elements and $\phi(n)$ generators, suppose that m is a given multiple of $\phi(n)$ For any pair of generators x, y of C there is exactly one element $e_m(x,y) \in U_1\mathbf{Z}C$ such that

$$(3) \qquad (x-1)^m = e_m(x,y)\cdot(y-1)^m .$$

In fact, we need only put $e_m(x,y) = (1+y+\cdots+y^{a-1})^m - b(1+y+\ldots y^{n-1})$, where $x = y^a$ and $a^m = 1+bn$. As they were first explored by BASS [1], we shall refer to these special units as BASS units of level m in $\mathbf{Z}C$. From the defining equation (3) it is obvious that they satisfy the relations

$$(4)\,(\mathrm{i})\;\; e_m(x,y)e_m(y,z) = e_m(x,z) \quad \text{and} \quad (\mathrm{ii})\;\; e_m(x^{-1},y) = e_m(x,y) ,$$

provided, for the sake of (ii), that m is also a multiple of n and hence $(x^{-1}-1)^m = (x-1)^m$. For $n > 2$, these relations quickly lead to the conclusion that the group $B_m(C)$ generated by the $e_m(x,y)$ has rank $\leq$ $\phi'(n) = \frac{1}{2}\phi(n) - 1$. To round out the picture, let us put $\phi'(1) = \phi'(2) = 0$.

Using an essentially analytic independence theorem for cyclotomic units, BASS [1] shows that $B_m(C)$ is free of rank exactly $= \phi'(n)$. In other words, (i) and (ii) generate all the relations between the BASS units. Not only that: as S ranges over all subgroups of C, the product of the groups $B_m(S) \subset U_1\mathbf{Z}C$ is *direct*, and hence (by a rank count) of finite index.

Turning back to an arbitrary finite abelian group A, we can now invoke a result of MILNOR (cf. [2], Ch.XI, Theorem 7.1.c), which says that the product of $U_1\mathbf{Z}C$, as C ranges over all cyclic subgroups of A, has finite index in $U_1\mathbf{Z}A$. Letting n be the order of a maximal cyclic subgroup of A, and fixing a multiple m of n and $\phi(n)$, we can do yet another rank count to deduce the *Theorem of Bass-Milnor: the product*

$$(5) \qquad M_m(A) = \prod_{C \subseteq A} B_m(C) \subset U_1\mathbf{Z}A$$

is direct and of finite index.

In this general context, the level m of the BASS units could actually be allowed to vary with C, as long as it is divisible by both $|C|$ and $\phi(|C|)$. For our purposes, however, working with a fixed m has certain advantages —cf. equation (6) below.

2. Results

If $\psi : A \to K^\times$ is a character of order q as above, and if $\psi(x)$ has order k for some $x \in A$, the factorization of the polynomial $X^k - 1$ shows that $N_\psi(x-1)^m = (x^k-1)^{\ell m}$, where $q = k\ell$. As a consequence, we get

$$N_\psi e_m(x,y) = e_m(x^k, y^k)^\ell \ , \tag{6}$$

whenever x and y generate the same subgroup C of A. It is important to note that $C^k \subseteq A_\psi = \ker(\psi)$, so that (6) implies

$$N_\psi : M_m(A) \to M_m(A_\psi) \ . \tag{7}$$

This fact will be crucial to the proof of our main result, which we now state.

Theorem. *The map $Q_\psi : u \mapsto u^q / N_\psi(u)$ defines an injection of finite index from the direct product*

$$M_{m,\psi}(A) = \prod_{\psi(C) \neq 1} B_m(C) \tag{8}$$

into the group of ψ-normal units in $U\mathbf{Z}A$.

PROOF. By splitting the index set $\{C \subseteq A\}$ of equation (5) into two parts, namely $\{\psi(C) \neq 1\}$ and $\{C \subseteq A_\psi\}$, we obtain $M_m(A)$ as the direct product $M_{m,\psi}(A) \times M_m(A_\psi)$. Therefore $1 \neq u \in M_{m,\psi}(A)$ implies $1 \neq u^q \cdot N_\psi(u^{-1}) = Q_\psi(u)$, on account of (7) — proving injectivity. Of course, any element of the form $Q_\psi(u)$ is ψ–normal.

Moreover, (6) shows that Q_ψ is trivial on $M_m(A_\psi)$, and hence the Q_ψ-image of $M_m(A)$ equals that of $M_{m,\psi}(A)$. In particular, the latter has finite index in $Q_\psi(U\mathbf{Z}A)$. Now, $N_\psi(u) = 1 \Longrightarrow u^q = Q_\psi(u)$, and thus a suitable power of u will lie in $Q_\psi(M_{m,\psi}(A))$.

Remarks. 1. It would be easy, though notationally tedious, to write down an explicit basis for $Q_\psi(M_{m,\psi}(A))$ in terms of Bass units. In the notation of (6), one could take elements of the form

$$e_m(x,y)^q \cdot e_m(x^k, y^k)^{-\ell} \quad , \tag{9}$$

where y is a fixed generator of a $C \subseteq A$ with $\psi(C) \neq 1$. In view of (4), one would select only one element from each pair x, x^{-1} generating the same C but not containing y. Each C contributes $\phi'(|C|)$ basis elements, giving the group of ψ-normal units a free rank of

$$\mathrm{rk}\ (\ker N_\psi) = \sum_{\psi(C) \neq 1} \phi'(|C|). \tag{10}$$

2. For $q = 2$, a unit is called ψ-unitary, if $u \cdot \tilde{\psi}(u^*) = \pm 1$, where $*$ denotes the involution defined on $\mathbf{Z}A$ by $x \mapsto x^{-1}$ for $x \in A$. By part (ii) of (4), we have $u = u^*$ for all $u \in M_m(A)$.

Thus $u \in M_m(A)$ is ψ-unitary if and only if u^2 is ψ-normal, and hence $Q_\psi(M_{m,\psi}(A))$ is also of finite index in the group of ψ-unitary units of $U\mathbf{Z}A$. This is the gist of Bovdi's Theorem (cf. [3], p. 472), which describes a basis $e_m(x,y)^2 e_m(x^2,y^2)^{-1}$ *à la* (9), and determines the rank *à la* (10), in the cyclic case. Indeed, if A is cyclic of order $n = 2^s t$, with $s > 0$ and t odd, the subgroups $C \subseteq A$ on which ψ is non-trivial correspond to the divisors $d \mid n$ divisible by 2^s. Hence the free rank of the ψ-unitary subgroup of $U\mathbf{Z}A$ equals

$$\sum_{d>2} \left(\frac{\phi(d)}{2} - 1 \right) \quad , \tag{11}$$

as d runs over these divisors.

References

[1] H. Bass, The Dirichlet unit theorem, induced characters, and Whitehead groups of finite groups, *Topology* **4** (1966), 391–410.
[2] H. Bass, Algebraic K-Theory, Benjamin, New York, 1968.
[3] A.A. Bovdi, Unitarnaya podgruppa multiplikativnoi gruppy celočislennogo gruppovogo kolca cykličeskoi gruppy, *Matem. Zametki* **41** (1987), 469–474.

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