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A counterexample on completeness in relator spaces

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Abstract. Disproving a plausible conjecture on completeness of generalized uniformities, we show that strong topological directedness cannot replace strong proximal directedness in a condition sufficient for comleteness (defined by the convergence or adherence of not necessarily directed Cauchy nets).

Preliminaries

Let us recall some definitions from [2, 3]. A *relator* $\mathcal{R}$ on the set X is a nonvoid family of reflexive relations on X. A (not necessarily directed) net (x_α) *converges* (*adheres*) to $x \in X$ if (x_α) is eventually (frequently) in $R(x)$ for each $R \in \mathcal{R}$. A net (x_α) is *convergence* (*adherence*) *Cauchy* if it is convergent (adherent) with respect to each relator $\{R\}$ ($R \in \mathcal{R}$). The relator $\mathcal{R}$ is *convergence* (*adherence*) *complete* if each convergence (adherence) Cauchy net is convergent (adherent). $\mathcal{R}$ is *topologically transitive* if for each $x \in X$ and $R \in \mathcal{R}$ there are $S, T \in \mathcal{R}$ such that $T(S(x)) \subset R(x)$; *strongly proximally directed* if for any $n \in \mathbf{N}$, $A_i \subset X$ and $R_i \in \mathcal{R}$ $(1 \le i \le n)$ there is an $R \in \mathcal{R}$ such that $R(A_i) \subset R_i(A_i)$ $(1 \le i \le n)$; *topologically directed* if for each $x \in X$, $\{R(x) : R \in \mathcal{R}\}$ is a filter base; *topologically compact* provided that if an $R_x \in \mathcal{R}$ is assigned to each $x \in X$ then there is a finite $A \subset X$ such that the sets $R_x(x)$ $(x \in A)$ cover X. [3] Corollary 3.3 states that

Theorem A. *Any strongly proximally directed, topologically transitive, topologically compact relator is convergence as well as adherence complete.*

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Strong proximal directedness falls here out of line: one would expect the theorem to hold for topologically directed relators. This problem was raised by ÁRPÁD SZÁZ in a talk presented at Drujba (Bulgaria) in 1990. [His question was in fact somewhat different, see Remark 4).] We are going to show that even strong topological directedness cannot replace the strong proximal directedness. (The definition is obvious: $\mathcal{R}$ is *strongly topologically directed* if for any $n \in \mathbf{N}$, $x_i \in X$ and $R_i \in \mathcal{R}$ $(1 \le i \le n)$ there is an $R \in \mathcal{R}$ such that $R(x_i) \subset R_i(x_i)$ $(1 \le i \le n)$.)

The example

On $X = \{-2,-1,0\} \cup \mathbf{N}$, let $\mathcal{R} = \{R_n : n \in \mathbf{N}\}$, where

$$R_n(0) = \{0\} \cup \{k \in \mathbf{N} : \ k > n\} \qquad (n \in \mathbf{N}),$$
$$R_n(n) = \{-2,-1,n\} \qquad (n \in \mathbf{N}),$$

and $R_n(x) = \{x\}$ otherwise.

$\mathcal{R}$ is strongly topologically directed. Given $m \in \mathbf{N}$, $x_k \in X$ and $R_{n_k} \in \mathcal{R}$ $(1 \le k \le m)$, choose $n \in \mathbf{N}$ such that $n \ge n_k$ and $n \ne x_k$ $(1 \le k \le m)$; now $R_n(x_k) \subset R_{n_k}(x_k)$ for each k.

$\mathcal{R}$ is topologically transitive. $R_n \circ R_n = R_n$ $(n \in \mathbf{N})$, so $\mathcal{R}$ is in fact strongly transitive in the sense of [2].

$\mathcal{R}$ is topologically compact, since each $R_n(0)$ is cofinite.

$\mathcal{R}$ is not convergence complete. The sequence $-1,-2,-1,-2,\dots$ is convergence (and adherence) Cauchy (since $-1,-2 \in R_n(n)$), but not convergent.

$\mathcal{R}$ is not adherence complete. The non-directed net consisting of the points -1 and -2 is adherence (and convergence) Cauchy, but not adherent.

Remarks

1) It follows from Theorem A that $\mathcal{R}$ cannot be strongly proximally directed. A direct proof: consider $R_1(A)$ and $R_2(\mathbf{N} \setminus A)$ where A consists of the even positive integers. However, $\mathcal{R}$ is *proximally directed*, i.e. $\{R(A) : R \in \mathcal{R}\}$ is a filter base whenever $\emptyset \ne A \subset X$.

2) It is natural that the non-adherent net in the example is not directed, since if $\mathcal{R}$ is topologically compact then each directed net is adherent ([4] 4.3).

3) For uniformities, the two notions of completeness considered above are strictly stronger than the usual completeness. Indeed, the uniformity in [1] II.50 is complete but not adherence complete (there are even directed

adherence Cauchy nets that are not adherent). On the other hand, the complete uniformity induced on $X = \mathbf{N}^2$ by the metric

$$d((k_1,n_1),(k_2,n_2)) = \begin{cases} 1/k & \text{if } k_1 = k_2 = k,\ n_1 \neq n_2\,, \\ 0 & \text{if } (k_1,n_1) = (k_2,n_2)\,, \\ 1 & \text{otherwise} \end{cases}$$

is not convergence complete: take the net $(x)_{x\in X}$ with the partial order $(k_1,n_1) \leq (k_2,n_2)$ iff $k_1 = k_2,\ n_1 \leq n_2$.

4) Theorem A is proved in [3] in two steps (Theorems 3.1 and 3.2):

Theorem B. *If $\mathcal{R}$ satisfies the assumptions of Theorem A then $\mathcal{R}$ is a Lebesgue relator.*

($\mathcal{R}$ is a *Lebesgue relator* provided that if an $R_y \in \mathcal{R}$ is assigned to each $y \in X$ then there is an $R \in \mathcal{R}$ such that for each $x \in X$ there is a $y \in X$ with $R(x) \subset R_y(y)$.)

Theorem C. *Any Lebesgue relator is both convergence and adherence complete.*

ÁRPÁD SZÁZ originally asked whether topological directedness is sufficient in Theorem B; our example shows that even strong topological directedness is not enough here. The following simplified version of the example also answers the original question: drop the point -2. To see that the relator is not Lebesgue, take $R_2(1)$ and $R_1(x)$ $(x \neq 1)$.

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