

The property of smallness up to a complemented Banach subspace

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Abstract. This article investigates locally convex spaces which satisfy the property of smallness up to a complemented Banach subspace, the *SCBS* property, which was introduced by Djakov, Terzioğlu, Yurdakul and Zahariuta. It is proved that a bounded perturbation of an automorphism on a complete barrelled locally convex space with the *SCBS* is stable up to a Banach subspace. New examples are given, and the relation of the *SCBS* with the quasinormability is analyzed. It is proved that the Fréchet space l_{p+} does not satisfy the *SCBS*, therefore this property is not inherited by subspaces or separated quotients.

1. Introduction

Our terminology for locally convex spaces is standard and we refer the reader to [11] or [8]. For a locally convex space X , $U(X)$ denotes a basis of absolutely convex neighborhoods of the origin in X and for $U \in U(X)$, p_U is the gauge of U . A linear operator T from a locally convex space X into another Y is bounded if $T(U)$ is a bounded subset of Y for some $U \in U(X)$.

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We say that a pair (X, Y) has *the bounded factorization property* and write $(X, Y) \in \mathcal{BF}$ if each continuous linear operator from X into X that factors over Y is bounded.

By GROTHENDIECK ([7], p. 107), a locally convex space X is said to be *quasi-normable* if for each $U \in U(X)$ there is $V \in U(X)$ so that for every $\epsilon > 0$ there exists a bounded subset A of X such that $V \subset A + \epsilon U$.

Following [5], we say that a locally convex space X satisfies *the property of smallness up to a complemented Banach subspace*, abbreviated from now on as the *SCBS* property, if for each bounded subset A of X , for each $U \in U(X)$, and for every $\epsilon > 0$, there are complementary subspaces B and E of X such that B is a Banach space and $A \subset B + \epsilon U \cap E$.

In [5], it was proved that all Banach-valued l_p -Köthe spaces have the *SCBS* property and the bounded perturbation of an automorphism on an l_p -Köthe space is stable up to some Banach basic subspace. This was essential there to get a modification of the generalized Douady lemma [16].

We still have no complete characterization of Fréchet or *DF*-spaces with the *SCBS* property. In this work we analyze the *SCBS* property, and show that more general form of Köthe spaces, say, l -Köthe spaces, some quasi-normable Fréchet spaces and the strong duals of some asymptotically normable Fréchet spaces have this property. We also get a characterization of quasi-normable l -Köthe spaces in terms of their basic Banach subspaces (see [2], [4]). Modifying Theorem 1 in [5] we obtain, as an extension of Proposition 3 there, that any infinite dimensional complemented Banach subspace of a c_0 -Köthe space is basic. This result may be considered as a partial answer to the well-known Pelczynski problem: Does a complemented subspace of a space with basis have a basis? Moreover, in this case we confirm the conjecture of BESSAGA [1] that each complemented subspace of a Köthe space is isomorphic to a basic subspace.

2. Results

Note that a locally convex direct sum of Banach spaces has the *SCBS* property trivially, since every bounded subset of it is contained in a finite sum of its components.

We follow DRAGILEV [6] to define l -Köthe spaces. First, we need:

Definition. Let $(X, \|\cdot\|)$ be a Banach space with a basis (x_n) and let (x'_n) denote the sequence of coefficient functionals. The norm $\|\cdot\|$ of X is called *monotonous* if the following implication holds: for any $x, y \in X$, $|x'_n(x)| \leq |x'_n(y)|$ ($\forall n \in \mathbb{N}$) implies $\|x\| \leq \|y\|$ (see also [9]).

It is known that every Banach space with an unconditional basis has a monotonous norm which is equivalent to its original norm. Indeed, it is enough to put

$$\|x\| = \sup_{|\alpha_k| \leq 1} \left| \sum_k x'_k(x) \alpha_k x_k \right|$$

where $|\cdot|$ denotes the original norm.

Throughout this work we denote by l a Banach sequence space in which the canonical system (e_n) is an unconditional basis, with a monotonous norm $\|\cdot\|$ satisfying $\|e_n\| = 1$ for each $n \in \mathbb{N}$. Let Λ be the class of all such spaces; in particular, l_p and c_0 are in this class.

Definition. Let $l \in \Lambda$ and $\|\cdot\|$ be a norm (monotonous in this work) in l . If $(a_{n,p})$ is a Köthe matrix, then the l -Köthe space $\lambda^l(a_{n,p})$ is the locally convex space of all sequences (t_n) , such that $(t_n a_{n,p}) \in l$ for any $p \in \mathbb{N}$, with the topology generated by the seminorms

$$|(t_n)|_p = \|(t_n a_{n,p})\|, \quad p \in \mathbb{N}.$$

Note that $|e_n|_p = a_{n,p} \|e_n\| = a_{n,p}$.

Proposition 1. *Every l -Köthe space has the SCBS property. In particular, every c_0 -Köthe space has the SCBS property.*

PROOF. Let $X = \lambda^l(a_{n,k})$ be a l -Köthe space. Let A be a bounded subset of X . Without loss of generality we may assume that

$$A = \left\{ x \in X : |x|_k = \|(x_i a_{i,k})\| = \sup_{|\alpha_i| \leq 1} \left| \sum_i \alpha_i x_i a_{i,k} e_i \right| \leq c_k \forall k \right\}$$

where (c_k) is a sequence of real numbers increasing to ∞ . Choose (c_k) so that $(\frac{a_{i,k}}{c_k})_k$ tends to zero for all i . For this purpose, it is enough to replace those c_k 's with $k \geq i$ by bigger ones, for example by $k \max_{1 \leq i \leq k} a_{i,k}$. Here $\|\cdot\|$, $|\cdot|$ respectively denote the monotonous and the original norm in l .

We set $\gamma_i = \sum_k \frac{a_{i,k}}{2^k c_k}$. Then, for any $x \in A$

$$\begin{aligned} \left| \sum_i \alpha_i \gamma_i x_i e_i \right| &= \left| \sum_i \alpha_i x_i \left(\sum_k \frac{a_{i,k}}{2^k c_k} \right) e_i \right| \\ &= \left| \sum_k \frac{1}{2^k} \left(\sum_i \alpha_i \frac{a_{i,k}}{c_k} x_i e_i \right) \right| \leq 1 \end{aligned}$$

or $\|(x_i \gamma_i)\| \leq 1$ holds.

Fix $\epsilon > 0$, $k_0 \in \mathbb{N}$ and set $B = [e_i : \epsilon \gamma_i \leq a_{i,k_0}]$, $E = [e_i : \epsilon \gamma_i > a_{i,k_0}]$ where the square brackets denote the closed linear span of the corresponding vectors. For $x \in B$ and $k \in \mathbb{N}$, since

$$|x_i a_{i,k_0}| \geq |x_i \gamma_i \epsilon| \geq \left| x_i \epsilon \frac{a_{i,k}}{2^k c_k} \right|$$

we have

$$\begin{aligned} \frac{1}{\epsilon} 2^k c_k \|(x_i a_{i,k_0})\| &\geq \frac{1}{\epsilon} 2^k c_k \|(x_i \gamma_i \epsilon)\| \\ &\geq \frac{1}{\epsilon} 2^k c_k \left\| \left(x_i \epsilon \frac{a_{i,k}}{2^k c_k} \right) \right\| = |x|_k, \end{aligned}$$

that is B is a basic Banach subspace.

If $x \in A \cap E$, since $|x_i a_{i,k_0}| \leq |x_i \gamma_i \epsilon|$ we have

$$|x|_{k_0} = \|(x_i a_{i,k_0})\| < \|(\epsilon x_i \gamma_i)\| \leq \epsilon$$

which means that X has the *SCBS* property. $\square$

The following condition should be compared with the weak quasi-normability condition of Grothendieck (see for example [4]).

Definition. A Fréchet space X satisfies *the condition (QN)* if there is a map $\pi \in \mathbb{N}^{\mathbb{N}}$ such that for every $p \in \mathbb{N}$ and for every $\epsilon > 0$ there is a complemented Banach subspace B of X on which the topology given by p -th norm and the original topology coincide and

$$U_{\pi(p)} \subset B + \epsilon U_p \tag{1}$$

holds.

Proposition 2. *Every quasi-normable l -Köthe space X satisfies the condition (QN).*

PROOF. $|\cdot|_p^*$ denotes the gauge functional of U_p° i.e.

$$|x'|_p^* := \sup \{ |x'(x)| : x \in X, |x|_p \leq 1 \}, \quad x' \in X'.$$

Assume $a_{j,p} \neq 0$. Since $\|\cdot\|$ is monotonous, for any $x \in U_p$ one has

$$|e'_j(x)| = |x_j| = \|x_j e_j\| \leq \left\| \left(\frac{1}{a_{j,p}} x_i a_{i,p} \right)_i \right\| = \frac{1}{a_{j,p}} |x|_p \leq \frac{1}{a_{j,p}}.$$

Since the coefficient functionals e'_i are continuous, we have

$$1 = |e'_i(e_i)| \leq |e'_i|_p^* a_{i,p}$$

from which $|e'_j|_p^* = \frac{1}{a_{j,p}}$ follows.

By [10], Theorem 7, since X is quasi-normable, there exists a strictly increasing function ϕ such that for all $p \in \mathbb{N}$ there is $q \in \mathbb{N}$ such that for all $k \in \mathbb{N}$ there exists $c > 0$ such that

$$|e'_i|_q^* \leq c\phi\left(\frac{|e'_i|_p^*}{|e'_i|_q^*}\right) |e'_i|_k^*$$

or $\frac{a_{i,k}}{a_{i,q}} \leq c\phi\left(\frac{a_{i,q}}{a_{i,p}}\right)$ holds for each i .

Fix any $p \in \mathbb{N}$. Then there exists $q = \pi(p) \in \mathbb{N}$ such that the above property holds. For the given $\epsilon > 0$, we set

$$I_1 = \{i : \epsilon a_{i,q} \leq a_{i,p}\}, \quad B = [e_i : i \in I_1],$$

where the square bracket denotes the closed linear span of the corresponding vectors. Then obviously $\epsilon|x|_q = \epsilon\|(x_i a_{i,q})\| \leq \|(x_i a_{i,p})\| = |x|_p$ for any $x \in B$, since $|\epsilon x_i a_{i,q}| \leq |x_i a_{i,p}|$ and the norm is monotonous. On the other hand for any $k > q$

$$\frac{a_{i,k}}{a_{i,q}} \leq c\phi\left(\frac{a_{i,q}}{a_{i,p}}\right) \leq c\phi\left(\frac{1}{\epsilon}\right)$$

for every $i \in I_1$, so we have $|x|_k \leq c\phi(\frac{1}{\epsilon})|x|_q$ for any $x \in B$. Hence B is a Banach space whose topology is given by the p -th norm.

Let $E = [e_i : i \notin I_1]$; then $X = B \oplus E$ and $\epsilon|x|_q > |x|_p$ for $x \in E$, therefore the condition (QN) holds. $\square$

Further, the condition (QN) characterizes quasinormability in terms of basic Banach subspaces in l -Köthe spaces (see [2], Proposition 3.3 and [4], Corollary 8). Indeed:

Proposition 3. *If X satisfies the condition (QN) then X is a quasinormable Fréchet space.*

PROOF. Since $\pi(p) > p$, $U_{\pi(p)} \subset U_p$ and condition 1 implies that $U_{\pi(p)} \subset A + \epsilon U_p$ where A is a ball in B with radius $c(1 + \epsilon)$. In fact, when $x \in U_{\pi(p)}$ we can write $x = b + \epsilon u$ for some $b \in B$ and $u \in U_p$. Let $|\cdot|_B$ denote the norm of B . Then $|b|_B \leq c|b|_p \leq c(|x|_p + |\epsilon u|_p) \leq c(1 + \epsilon)$ since p -th norm and the space X induce the same topology on the Banach space B . Hence X is a quasi-normable Fréchet space. $\square$

Proposition 4. *Every Fréchet space X with the condition (QN) has the SCBS property.*

PROOF. Let A be a bounded subset in X and $\pi(p)$ be as in condition 1. Find $\rho > 0$ such that $A \subset \rho U_{\pi(p)}$. Then by assumption it follows that $A \subset \rho U_{\pi(p)} \subset B + \rho \epsilon U_p \cap E$. $\square$

Definition. A Fréchet space X is said to satisfy the condition (AN) if for every $q \in \mathbb{N}$ there is $r \in \mathbb{N}$ so that for each $\epsilon > 0$ there exist two subspaces B and E of X (B is Banach and $X = B \oplus E$) such that

$$U_q^\circ \subset B' + \epsilon U_r^\circ \cap E'. \quad (2)$$

Here B' denotes the dual of B .

Proposition 5. *If X is a Fréchet space satisfying the condition (AN) , then X is an asymptotically normable Fréchet space.*

PROOF. Suppose X has the condition (AN) . Since $U_q^\circ \subset U_r^\circ$ for $r > q$, condition 2 implies that $U_q^\circ \subset B' \cap (1 + \epsilon)U_r^\circ + \epsilon U_r^\circ$. Since U_r° is weakly relatively compact, U_r° is strongly bounded. Hence $B' \cap (1 + \epsilon)U_r^\circ$ is strongly bounded or bounded in the norm topology of B' . So $U_q^\circ \subset MB_{B'} + \epsilon U_r^\circ$ holds for some $M > 0$, where $B_{B'}$ denotes the ball of B' . Of course this implies that X has DN_ϕ (in polarized form) i.e. X is asymptotically normable [13]. $\square$

Proposition 6. *The strong dual of a Fréchet space with the condition (AN) has the SCBS property.*

PROOF. Let X be a Fréchet space with the condition (AN) and let A' be a bounded subset of X' . Then there exists a $q > 0$ such that $A' \subset U_q^\circ$. By assumption, for this q , we find an $r > 0$ such that condition 2 holds. Also for an arbitrary zero-neighborhood C_α° of X' (where C_α is a bounded subset of X) there exists $s_\alpha > 0$ such that $C_\alpha \subset s_\alpha U_r$ and thus $U_r^\circ \subset s_\alpha C_\alpha^\circ$. Now apply condition 2 for $\frac{\epsilon}{s_\alpha}$ to obtain B' and E' depending on ϵ and α such that $U_q^\circ \subset B' + \frac{\epsilon}{s_\alpha} U_r^\circ \cap E'$ or $A' \subset U_q^\circ \subset B' + \epsilon C_\alpha^\circ \cap E'$. Hence X' has the *SCBS* property. $\square$

The following example shows that the *SCBS* property is not a necessary condition for the stability of a bounded perturbation of an automorphism up to some Banach subspace. Moreover it is not inherited by subspaces or quotients:

Example [12]. For $1 \leq p < \infty$, let $l_{p+} = \cap_{q>p} l_q = \cap_k l_{p_k}$, where $p_k \downarrow p$ (projective limit of l_p spaces). The topology of the Fréchet space l_{p+} may be then represented by means of the p_k -norms:

$$\|(x_n)\|_k = \left(\sum_n |x_n|^{p_k} \right)^{\frac{1}{p_k}}.$$

This space has the following properties:

- (i) It is a reflexive quasi-normable Fréchet space.
- (ii) Since $l_p \subset l_{p+} \subset l_q$ for all $q > p$, with continuous inclusions that are not compact, l_{p+} is not a Montel space.
- (iii) Since the canonical inclusions from $l_{p_{k+1}}$ into l_{p_k} are strictly singular and strictly cosingular, l_{p+} can have no infinite dimensional Banach subspace or quotient.

Since the canonical linking maps are strictly singular, we observe that any linear bounded operator T from l_{p+} to l_{p+} is also super strictly singular and hence $I + T$ is Frédhholm ([15], [14]).

If l_{p+} had the *SCBS* property, then for a given bounded subset A , $k > 0$ and $\epsilon > 0$, by assumption and (iii), there would exist a finite dimensional subspace B of l_{p+} such that:

$$A \subset B + \epsilon U_k,$$

that is A is precompact. Hence l_{p+} would be Montel, which contradicts (ii). Therefore, l_{p+} does not have the *SCBS* property and consequently it can not satisfy the condition (QN) .

Proposition 7. *The SCBS property does not pass to quotients.*

PROOF. l_{p+} is a quasi-normable Fréchet space, and thus by MEISE–VOGT’s characterization [10] of quasinormability in Fréchet spaces, it is isomorphic to a quotient of an l_1 -valued generalized Köthe space which, by [5], has the *SCBS* property, but l_{p+} does not. $\square$

Proposition 8. *The SCBS property does not pass to subspaces.*

PROOF. l_{p+} is a countable projective limit of l_p -spaces and thus by Remark 24.5 in [11], it is isomorphic to a subspace of some countable product of Banach spaces. This product can be understood as a Banach-valued Köthe space which, by [5], has the *SCBS* property. However, l_{p+} does not have. $\square$

One can follow the steps of the proof of Theorem 1 in [5] to get the following:

Theorem 9. *If X is a complete barrelled locally convex space with the SCBS property and T is a linear bounded (respectively, compact) operator on X into X , then there exist complementary subspaces B and E of X such that:*

- (i) B is a Banach (respectively, finite dimensional) space; and
- (ii) if π_E and i_E are the canonical projection onto E and embedding into X , respectively, then $1_E + \pi_E T i_E$ is an automorphism on E .

PROOF. Since T is a bounded operator there exists $U_0 \in U(X)$ such that $T(U_0)$ is a bounded set in X , therefore

$$\forall U \in U(X) \exists C_U > 0 : p_U(Tx) \leq C_U p_{U_0}(x).$$

Since X has the *SCBS* property there exist complementary subspaces B and E of X such that B is a Banach (respectively finite dimensional) space and $T(U_0) \subset B + \frac{1}{2}U_0 \cap E$. Therefore, setting $T_1 = \pi_E T i_E : E \rightarrow E$, we obtain that

$$p_{U_0}(T_1x) \leq \frac{1}{2}p_{U_0}(x) \quad \forall x \in E.$$

Now it is easy to see that the operator $1_E - T_1$ is an automorphism of E . Indeed, consider for any $x \in E$ the series

$$Sx = x + T_1x + T_1^2x + \dots + T_1^mx + \dots \quad (3)$$

It is convergent in E because, for any $U \in U(X)$, we have

$$p_U(T_1^m x) \leq C_U p_{U_0}(T_1^{m-1}x) \leq C_U \left(\frac{1}{2}\right)^{m-1} p_{U_0}(x), \quad m = 1, 2, \dots,$$

so by Banach–Steinhaus theorem, since X is barrelled and complete, the formula (3) defines a linear continuous operator $S : E \rightarrow E$.

Since $(1_E - T_1)Sx = S(1_E - T_1)x = x$, the operator S is inverse to the operator $1_E - T_1$. $\square$

Exactly in the same way as in ([5], Theorem 2), Theorem 9 enables us to have the following modification of the generalized Douady lemma in ([16], Section 6):

Theorem 10. *Suppose X_1 is a complete barrelled locally convex space with the SCBS property and X_2, Y_1, Y_2 are topological vector spaces. If $X_1 \times X_2 \simeq Y_1 \times Y_2$ and $(X_1, Y_2) \in \mathcal{BF}$, then there exists complementary subspaces E and B in X_1 and complementary subspaces of G and F in Y_1 such that B is a Banach space, $F \simeq E$ and $B \times X_2 \simeq G \times Y_2$.*

If, in addition, $(Y_1, X_2) \in \mathcal{BF}$, then G is a Banach space.

Proposition 11. *Since the space ϕ of finite sequences with the direct sum topology is a complete barrelled DF space, then in the above theorem X_1 can be taken to be ϕ and Y_2 to be either a Fréchet space or a locally convex space with the individual countable boundedness condition (see [3]).*

Since, by Proposition 1, c_0 -Köthe spaces have the SCBS property, we get:

Proposition 12. *Let X be a c_0 -Köthe space, and F, G be complementary subspaces in X , i.e. $X = F \oplus G$. If G is an infinite dimensional Banach space then $G \simeq c_0$, and moreover, F and G are isomorphic to some basic subspaces of X .*

PROOF. We have $X \times \{0\} \simeq F \times G$ and by Theorem 10 there exist complementary basic subspaces E and B in X and complementary subspaces

F_1 and G_1 in F such that B is a Banach space and

$$F_1 \simeq E, \quad B \simeq G_1 \times G.$$

Since every infinite dimensional basic Banach subspace of a c_0 -Köthe space is isomorphic to c_0 we obtain that $B \simeq c_0$. On the other hand, each infinite dimensional complemented subspace of c_0 is isomorphic to c_0 (see [9]), so G is isomorphic to c_0 . Finally, since $B \simeq c_0$, its complemented subspace G_1 is isomorphic to some basic subspace of B and $F \simeq E \oplus G_1$ is isomorphic to some basic subspace of X . $\square$

It would be interesting to have this result for l -Köthe spaces (see, for example [6], Proposition 2.2.1).

References

- [1] Cz. BESSAGA, Some remarks on Dragilev's theorem, *Studia Math.* **31** (1968), 307–318.
- [2] K. BIERSTEDT, R. MEISE and W. SUMMERS, Köthe sets and Köthe sequence spaces, in: Functional Analysis, Holomorphy and Approximation Theory, Vol. 71, *North-Holland Math. Studies*, 1982, 27–91.
- [3] J. BONET and A. GALBIS, The identity $L(E, F) = LB(E, F)$, tensor products and inductive limits, *Note Mat.* **9** (1989), 195–216.
- [4] J. BONET and J. DIAZ, On the weak quasinormability condition of Grothendieck, *Doğa Mat.* **15** (1991), 154–164.
- [5] P. DJAKOV, T. TERZIOĞLU, M. YURDAKUL and V. ZAHARIUTA, Bounded operators and isomorphisms of Cartesian products of Fréchet spaces, *Michigan Math. J.* **45** (1998), 599–610.
- [6] M. M. DRAGILEV, Bases in Köthe spaces, *Rostov State University*, 1983.
- [7] A. GROTHENDIECK, Sur les espaces (F) et (DF), *Summa Brazil Math.* **3** (1954), 57–122.
- [8] H. JUNEK, Locally convex spaces and operator ideals, *B.G. Teubner, Leipzig*, 1983.
- [9] J. LINDENSTRAUSS and L. TZAFRIRI, Classical Banach spaces I, *Springer, Berlin*, 1977.
- [10] R. MEISE and D. VOGT, A characterization of the quasi-normable Fréchet spaces, *Math. Nachr.* **122** (1985), 141–150.
- [11] R. MEISE and D. VOGT, Introduction to Functional Analysis, *Clarendon Press, Oxford*, 1997.
- [12] G. METAFUNE and V. B. MOSCATELLI, On the space $l_{p+} = \bigcap_{q>p} l_q$, *Math. Nachr.* **147** (1990), 7–12.

- [13] T. TERZIOĞLU and D. VOGT, On asymptotically normable Fréchet spaces, *Note Mat.* **XI** (1991), 289–296.
- [14] V. V. WROBEL, Streng singuläre Operatoren in lokalkonvexen Räumen, *Math. Nachr.* **83** (1978), 127–142.
- [15] V. V. WROBEL, Striktsinguläre Operatoren in lokalkonvexen Räumen II, Beschränkte Operatoren, *Math. Nachr* **110** (1983), 205–213.
- [16] V. P. ZAHARIUTA, On the isomorphisms of Cartesian products of locally convex spaces, *Studia Math.* **46** (1973), 201–221.

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