

A theorem on description adequacy.

In memory of my good friend, Tibor Szele.

By DAVID ELLIS in Santa Monica, California.

1. Introduction. There are six well-known propositions concerning the adequacy (defined in Section 3 below) of certain classes of nets [2], [3]¹⁾ in a topological space to describe the topology of the space. They may be stated :

(P1). The set of (countable) sequences in a locally separable space (and, hence, in particular, a metric space) adequately describes the topology of the space [5].

(P2). The set of phalanxes in a topological space adequately describes the topology of the space [4].

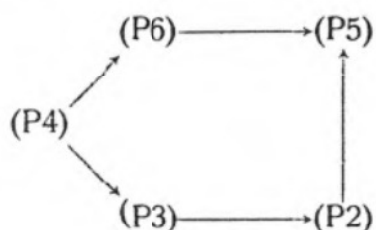
(P3). The set of topophalanxes in a topological space adequately describes the topology of the space [4].

(P4). The set of ultraphalanxes in a topological space adequately describes the topology of the space [4].

(P5). The set of all nets in a topological space adequately describes the topology of the space [2].

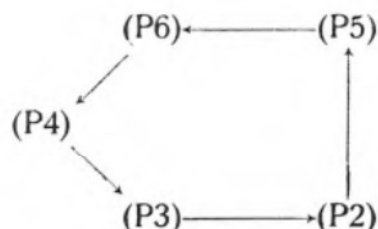
(P6). The set of ultranets in a topological space adequately describes the topology of the space [2].

(P1) is a trivial consequence of the axiom of choice. It is also clear that one has immediately the following implication diagramm :



¹⁾ Numbers in square brackets refer to the Bibliography concluding the note.

The implication $(P5) \rightarrow (P6)$ is a consequence of the proposition [2] that every net has a subnet which is an ultranet²⁾. In Section 2 below, it is shown that every net has a subnet which is a phalanx. Since every subnet of an ultranet is an ultranet, this yields the implication $(P6) \rightarrow (P4)$. Thus, one has the implication cycle:



so that all five propositions may be deduced from the validity of the simplest; namely, (P5).

Now, proofs of (P1) to (P6) all employ, explicitly or otherwise, some form of the axiom of choice. It is then a natural idea to construct an adequate class of nets in a topological space directly from the mapping whose existence is a form of the axiom of choice; namely, the selection operator. That this is possible, and, indeed, simple, is shown in Section 3 below.

2. By *topological space*, hereafter called simply *space* and denoted S , we shall mean a non-null T_1 space [1], [5].

A *directed set* is simply a partially ordered set in which each pair of elements (and, hence, each non-null finite subset) has an upper bound. A subset of a directed set is *residual* if it contains all elements above some given one. A subset of a directed set is *cofinal* if it contains an upper bound for every finite non-null subset of the directed set.

A *net* [2], [3] is any mapping of a directed set into a set. If $n: \mathcal{A} \rightarrow S$ is a net, one refers to the net n in S based on $\mathcal{A}$.

A *stack* [4] is the directed set composed of all non-null finite subsets of some set, the partial ordering being taken as set-theoretic inclusion.

A net based on a stack is called a *phalanx* [4].

If $n: \mathcal{A} \rightarrow S$ is a net and $T \subset S$ one says [3]:

1. n *decides for* T if n maps some residual subset of $\mathcal{A}$ into T .
2. n *decides against* T if n decides for $S - T$.
3. n *decides about* T if either 1. or 2. subsists.
4. n is *undecided about* T (or fails to decide about T) if neither 1. nor 2. subsists.

Clearly, 4. subsists if and only if both T and $S - T$ contain images under n of cofinal subsets of $\mathcal{A}$.

²⁾ KELLEY uses the term *universal net* rather than ultranet.

If $n: \mathcal{A} \rightarrow S$ and $m: \Gamma \rightarrow S$ are nets and if there is a net $\gamma: \Gamma \rightarrow \mathcal{A}$ which decides for every residual subset of $\mathcal{A}$ and if $\gamma n = m$, then m is called a *subnet* [2] of n .

Clearly, every subnet of a net decides for the same sets as does the net itself.

A net which decides about every subset of its range is called an *ultranet*.

A phalanx which is an ultranet is called an *ultraphalanx* [4].

If S is a space, a phalanx in S is called a *topophalanx* [4] if it decides about every open set.

If S is a space and $p \in S$ and n is a net in S , one says [2] that n *converges* to p , or that n has *limit* p , provided n decides for every open set of which p is a member.

Proposition. *Every net has a subnet which is a phalanx.*

PROOF. Let $n: \mathcal{A} \rightarrow S$ be a net. Consider the stack Γ of $\mathcal{A}$. Define $\gamma: \Gamma \rightarrow \mathcal{A}$ by ordinary induction as follows: If $\alpha \in \Gamma$ and α contains exactly one point, let $\alpha\gamma$ be that point. Suppose γ has been defined for all members of Γ which contain less than $n > 1$ elements. If $\xi \in \Gamma$ has exactly n elements, let $\xi\gamma$ be any upper bound of the (necessarily finite) set $\{\eta\gamma\}$ where η ranges over all non-null proper subsets of ξ . Define then $m = \gamma n$.

3. Let S be any non-null set and let $\mathfrak{S}$ be the set of all non-null subsets of S . A *selection operator* in S is a mapping $f: \mathfrak{S} \rightarrow S$ having the property that for each $T \in \mathfrak{S}$, $Tf \in T$. The existence of a selection operator for each non-null S is a rudimentary form of the axiom of choice.³⁾

If f is a selection operator in S and if $\mathfrak{A} \subset \mathfrak{S}$ and $\mathfrak{A}$ is directed by inverse inclusion (that is, the set-theoretic product of any two members of $\mathfrak{A}$ contains a member of $\mathfrak{A}$), then $f/\mathfrak{A}$ shall be called a *selector net* in S .

A class $\mathfrak{N}$ of nets in a space S is said to *adequately describe* the topology of S provided the following proposition is valid: If $T \subset S$ and $p \in S$, then p is an accumulation point of T if and only if there is a net in $T - \{p\}$ which is a member of $\mathfrak{N}$ and which has limit p .

Proposition. *If S is a space and if f is a selection operator in S , then the selector nets derived from f adequately describe the topology of S .*

PROOF. Let $T \subset S$ and $p \in S$. If there is any net, and, in particular, a selector net, in $T - \{p\}$ with limit p , then p is an accumulation point of S by (P5). Suppose, alternatively, that p is an accumulation point of T . Then $T - \{p\}$ is non-null; indeed, infinite. Let $\mathfrak{A}$ be the set $\{G \cap (T - \{p\})\}$ where G ranges over all open sets of which p is a member. Since $\mathfrak{A}$ is actually

³⁾ Cf. TIBOR SZELE, On Zorn's lemma, *Publ. Math. Debrecen* 1 (1950), 254–257 and GARRETT BIRKHOFF, *Lattice theory (Revised Ed.)*, New York, 1947.

a groupoid under set-theoretic product, $f/\mathfrak{A}$ is a selector net whose values lie in $T - \{p\}$. If $p \in G$ and G is open, $f/\mathfrak{A}$ maps $\{H \cap (T - \{p\})\}$ into G where H ranges over all open sets with $p \in H \subset G$. Thus, $f/\mathfrak{A}$ decides for G . Hence, $f/\mathfrak{A}$ converges to p .

Bibliography.

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