

Partitions into primes.

Dedicated to the memory of Tibor Szele.

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1. Introduction. Let $P(n)$ denote the number of partitions of the integer n into primes (1 is not counted as prime), repetitions being allowed. That is, $P(n)$ is the number of ways n can be expressed in the form $n_1 p_1 + n_2 p_2 + \dots$, where p_j denotes the j th prime number and $n_1, n_2, \dots$ are arbitrary non-negative integers. The purpose of this note is to prove that

$$(1) \quad P(n+1) \geq P(n) \quad (n = 1, 2, 3, \dots).$$

In another paper¹⁾ we have proved that, if A is any non-empty set of positive integers and $F_A(n)$ denotes the number of partitions of the integer n into parts taken from the set A , repetitions being allowed, then $F_A(n)$ is a non-decreasing function of n for large²⁾ positive n if and only if either (I) A contains the element 1 or (II) A contains more than one element and, if we remove any single element from A , the remaining elements have greatest common divisor 1. This result shows that (1) is true if n is sufficiently large, and other results in the same paper show that in fact $\lim_{n \rightarrow \infty} \{P(n+1) - P(n)\} = +\infty$. However, the methods employed there do not provide a good estimate of the point at which the monotonicity of $P(n)$ begins. In the present paper we prove (1) by using an argument particularly adapted to the case where A is the set of prime numbers.

Let $P_k(n)$ denote the number of partitions of the integer n into parts taken from the first k primes, repetitions being allowed. Thus we have the formal power-series relation

$$(2) \quad \sum_{n=0}^{\infty} P_k(n) X^n = \prod_{j=1}^k (1 - X^{p_j})^{-1}.$$

Since $P(n) = P_k(n)$ for $n < p_{k+1}$, the assertion (1) will be proved if we can establish the following by induction on k .

¹⁾ *Monotonicity of partition functions*, to be published in *Mathematika*.

²⁾ In case (I) obviously $F_A(n+1) \geq F_A(n)$ for all n .

If k is a positive integer greater than 2, then

(A_k) $P_k(n+1) \geq P_k(n)$ for any positive integer n , and

(B_k) $P_k(p+1) > P_k(p)$ for any prime number k greater than p_k .

2. The case $k=3$. In view of (2) we have the formal power-series relation

$$1 + \sum_{n=0}^{\infty} \{P_3(n+1) - P_3(n)\} X^{n+1} = \frac{1-X}{(1-X^2)(1-X^3)(1-X^5)} = \\ = \frac{1}{30(1-X)^2} + \frac{a}{1-X} + \sum_{\mu=1}^2 \frac{b_{\mu}}{1-e^{2\pi i \mu/3} X} + \sum_{\nu=1}^4 \frac{c_{\nu}}{1-e^{2\pi i \nu/5} X},$$

where $a, b_1, b_2, c_1, c_2, c_3, c_4$ are certain complex numbers which could be calculated but whose values we shall not require. Hence if n is a positive integer

$$P_3(n+1) - P_3(n) = \frac{n+2}{30} + a + \sum_{\mu=1}^2 b_{\mu} e^{2\pi i \mu(n+1)/3} + \sum_{\nu=1}^4 c_{\nu} e^{2\pi i \nu(n+1)/5},$$

and so

$$(3) \quad P_3(n+1) - P_3(n) = \left\lfloor \frac{n-1}{30} \right\rfloor + \psi(n),$$

where $[x]$ denotes the greatest integer not exceeding the real number x and ψ is a function on the positive integers which has period 30. The values of $\psi(n)$ can be most easily found by taking $n=1, 2, \dots, 30$ in (3). We find that

$$\psi(n) = \begin{cases} 0 & \text{if } n \equiv 0, 2, 3, 5, 6, 8, 10, 12, 15, 18, 20 \pmod{30}, \\ 2 & \text{if } n \equiv 19, 29 \pmod{30}, \\ 1 & \text{otherwise.} \end{cases}$$

Thus $\psi(n) \geq 0$ for all n and $\psi(n) \geq 1$ if $(n, 30) = 1$. Hence assertions (A₃) and (B₃) are valid.

3. Inductive step. Suppose k is a positive integer greater than 3 and assume that assertions (A_{k-1}) and (B_{k-1}) are valid. Then we shall show that (A_k) and (B_k) are valid.

We begin by remarking that if n is any integer

$$(4) \quad P_k(n) = P_{k-1}(n) + P_k(n - p_k) = \\ = P_{k-1}(n) + \begin{cases} 0 & \text{if } n < p_k \text{ or } n = p_k + 1, \\ 1 & \text{if } n = p_k, \\ P_k(n - p_k) & \text{if } n \geq p_k + 2. \end{cases}$$

This is equivalent to the formal power-series identity

$$(1 - X^{p_k}) \sum_{n=0}^{\infty} P_k(n) X^n = \sum_{n=0}^{\infty} P_{k-1}(n) X^n,$$

which is an immediate consequence of (2). Alternatively (4) can be established by noticing that the first term on the right is equal to the number of parti-

tions of n into parts taken from the first k primes in which p_k does not actually occur as a part, while the second term is equal to the number of partitions of n into parts taken from the first k primes in which p_k does actually occur as a part.

Now if $1 \leq n < p_k$, then $P_k(n+1) \geq P_{k-1}(n+1) \geq P_{k-1}(n) = P_k(n)$ by (A_{k-1}) and (4). If $n = p_k$, then $P_k(n+1) = P_k(p_k+1) = P_{k-1}(p_k+1) \geq P_{k-1}(p_k) + 1 = P_k(p_k) = P_k(n)$ by (B_{k-1}) and (4). If $n > p_k$ and if we have proved that $P_k(m+1) \geq P_k(m)$ for $m = 1, 2, \dots, n-1$, then

$$P_k(n+1) = P_{k-1}(n+1) + P_k(n+1-p_k) \geq P_{k-1}(n) + P_k(n-p_k) = P_k(n)$$

by (A_{k-1}) and (4). Hence $P_k(n+1) \geq P_k(n)$ for all positive integers n and so assertion (A_k) is proved.

Now suppose p is a prime number greater than p_k . Then

$$P_{k-1}(p+1) > P_{k-1}(p) \quad \text{by } (B_{k-1}) \quad \text{and} \quad P_k(p+1-p_k) \geq P_k(p-p_k)$$

by (A_k) . Hence by (4)

$$P_k(p+1) = P_{k-1}(p+1) + P_k(p+1-p_k) > P_{k-1}(p) + P_k(p-p_k) = P_k(p).$$

Thus (B_k) is established and our proof is complete.

(Received September 5, 1955.)