On the Lebesgue property in uniform spaces.

To the memory of Professor Tibor Szele.

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This paper is mainly concerned with what might be called the finite Lebesgue property. A. A. MONTEIRO and M. M. PEIXOTO ([7], [8]) have introduced the notion of Lebesgue number in metric space and have discussed the relations between the Lebesgue number and uniform continuity of continuous functions on it.[1])

As far as possible, we use the notations and notions of P. SAMUEL [9]. All spaces considered will be assumed to be separated uniform spaces.

The following definitions and notations are used throughout the paper.

Let E be a separated uniform space. A *covering* of E is a family of open sets whose union is E. A covering is called *binary* if it consists of two open sets or *finite* if it consists of a finite family of open sets.

Let $\mathfrak{S}$ be a filter of the diagonal of $E \times E$ (a filter of surroundings of E). If $A \subset E$ and $V \in \mathfrak{S}$, we denote by $V(A)$ the image of the set $(E \times A) \cap V$ by the projection of $E \times E$ onto the first factor E.

We shall say that a (finite) covering $\mathfrak{F} = \{O_\alpha\}$ of E has *(finite) Lebesgue property* if there is a surrounding V of $\mathfrak{S}$ such that, for every x of E, we can find an open set O_α satisfying $V(x) \subset O_\alpha$.

We shall say that a uniform space E has the *(finite) Lebesgue property,* if any (finite) covering of E has the (finite) Lebesgue property.

Lemma. *If any binary covering of the uniform space E has the Lebesgue property, then E has the finite Lebesgue property.*

PROOF. We shall prove first that E is normal.

Let F_1, F_2 be two disjoint closed sets, and $O_i = E - F_i$ $(i = 1, 2)$. Then $\{O_1, O_2\}$ is a binary covering of E. Therefore, since $\{O_1, O_2\}$ has the Lebesgue property, there is a surrounding $V \in \mathfrak{S}$ such that $V(x) \subset O_1$, or $V(x) \subset O_2$ for any x of E. For the surrounding V, we can find a symmetric surrounding W such that $W \circ W \subset V$. We shall show that $W(F_1) \cap W(F_2) = 0$. To prove

[1]) Generalizations of these results are treated in my papers [3], [4] and [5].

it, suppose that $W(F_1) \cap W(F_2) \neq 0$, and let x be a point of this set. Then there exist two points a_1, a_2 such that $(x, a_1) \in W$, $(x, a_2) \in W$ and $a_i \in F_i$ $(i=1,2)$. Therefore $(a_1, a_2) \in W \circ W \subset V$. On the other hand, since $a_1 \in F_1$, $a_1 \notin O_1$ and we have $V(a_1) \subset O_2$. Since $a_2 \in F_2$, $a_2 \notin O_2$ and we have $a_2 \notin V(a_1)$. This shows that $(a_1, a_2) \notin V$, which gives a contradiction. Thus we have proved the normality of E.

Now we shall prove that E has the finite Lebesgue property. Let $\{O_i\}$ $(i=1,2,\dots,n)$ be a finite covering of E. Since E is normal, $\{O_i\}$ is shrinkable and we can find a covering $\{G_i\}$ such that $G_i \subset \bar{G}_i \subset O_i$ $(i=1,2,\dots,n)$. Let $H_i = E - \bar{G}_i$, then, for each i, $\{O_i, H_i\}$ is a binary covering of E. Hence, by the assumptions of our Lemma, it has the Lebesgue property. There exists a surrounding $V_i \in \mathfrak{S}$ such that $V_i(x) \subset H_i$ or $V_i(x) \subset O_i$ for any x of E. Let $V = \bigcap_{i=1}^{n} V_i$, then, if $V(x) \subset H_i$ $(i=1,2,\dots,n)$, we have

$$V(x) \subset \bigcap_{i=1}^{n} H_i = \bigcap_{i=1}^{n} (E - \bar{G}) = E - \bigcap_{i=1}^{n} \bar{G}_i = \text{empty},$$

which contradicts the fact that $V(x)$ is not empty. Hence there is an index ι such that $V(x) \subset O_i$, and thus the proof is completed.

Theorem 1. *A uniform space E has the finite Lebesgue property, if and only if E is normal and every bounded continuous function on E is uniformly continuous.*

PROOF. Suppose that E has the finite Lebesgue property. By our Lemma E is normal. Let $f(x)$ be a bounded continuous function and ε a given positive number. Since $f(x)$ is bounded, the range of $f(x)$ is contained in a finite interval $[-\alpha, \alpha]$. Let $\{I_i\}$ $(i=1,2,\dots,n)$ be a finite covering of $[-\alpha, \alpha]$ consisting of open intervals with length less than ε. Putting $O_i = f^{-1}(I_i)$ $(i=1,2,\dots,n)$, $\{O_i\}$ is a finite covering of E, hence it has the Lebesgue property. Therefore we can find a surrounding $V \in \mathfrak{S}$ such that $V(a) \subset O_i$ for some index i depending of a. Hence from $x, y \in V(a)$,

$$|f(x) - f(y)| \leqq |f(x) - f(a)| + |f(a) - f(y)| < 2\varepsilon.$$

This shows that any bounded continuous function is uniformly continuous.

The proof of the "if" part of Theorem 1 is contained in my paper [4], therefore we shall omit it here.

We shall consider the Lebesgue property of a uniform space (not necessarily finite). An important contribution is due to S. KASAHARA [6]. He proved the following

THEOREM. *If a uniform space E has the Lebesgue property, then E is paracompact.*

Let $\{O_\alpha\}$ be a given covering of E, then we can find a surrounding V for $\{O_\alpha\}$.

Let W be a surrounding such that $W \circ W \circ W \subset V$, then it is easily seen that the covering $\{W(x) | x \in E\}$ is a star refinement of the covering $\{O_\alpha\}$. Thus E is fully normal in the sense of J. W. TUKEY. Thus E is paracompact. This is the idea of the proof by S. KASAHARA.

On the other hand, J. DE GROOT and H. DE VRIES [1] (or K. ISÉKI [2]) proved the following.

THEOREM. *A locally metrizable, paracompact T_2-space is metrizable.*

This theorem follows from a result of the present author [2].

Therefore, we can easily prove the following

Theorem 2. *A uniform space E having the Lebesgue property is metrizable, if and only if E is locally metrizable.*

By a theorem of A. A. MONTEIRO and M. M. PEIXOTO [8], we can conclude

Theorem 3. *If a uniform space has the Lebesgue property, then every continuous function is uniformly continuous.*

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