On the volume of a polyhedron in non-Euclidean spaces.

To the memory of the geometer and the true friend I. Szele.

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It is known[1]) that the volume V of a convex Euclidean polyhedron containing a sphere of radius r and having f faces, e edges and v vertices satisfies the inequality

$$V \geqq \frac{e}{3} \sin \frac{\pi f}{e} \left(\tan^2 \frac{\pi f}{2e} \tan^2 \frac{\pi v}{2e} - 1 \right) r^3,$$

with equality only for the regular polyhedra circumscribed about the sphere. In the present paper we shall give an analogous estimating formula in non-Euclidean spaces. Our formula involves the volume of a quadrirectangular spherical or hyperbolic tetrahedron[2]) investigated by C. F. GAUSS (Cubirung der Tetraeder, [5], p. 228), J. BOLYAI ([13], pp. 105—115), L. SCHLÄFLI ([9], [10], [11] and [12]), N. I. LOBATSCHEFSKY [7], H. W. RICHMOND [8], H. S. M. COXETER [1], H. KNESER [6] and others. So the present investigation can be considered as a new treatment of this classical problem.

The above inequality, and the corresponding inequality in non-Euclidean spaces, can be derived from the following more general theorem [3]: Decompose the surface S of a sphere by a net N having v vertices and e edges into $f \geqq 4$ convex spherical polygons $S_1, \ldots, S_f$. Further, let $P_1, \ldots, P_f$ be f points of S and $g(x)$ a strictly increasing function defined for[3]) $0 \leqq x < \frac{1}{2}\sqrt{\pi S}$. Then

$$\sum_{i=1}^{f} \int_{S_i} g(P_i P) dS \geqq 4e \int_{\Delta} g(AP) dS,$$

where dS denotes the area element of S at the variable point P, and Δ a

[1]) Cf. [3] — Numbers in brackets refer to the bibliography at the end of this paper.

[2]) The word "quadrirectangular", used by H. S. M. COXETER [2], indicates the fact that all four faces are right triangles.

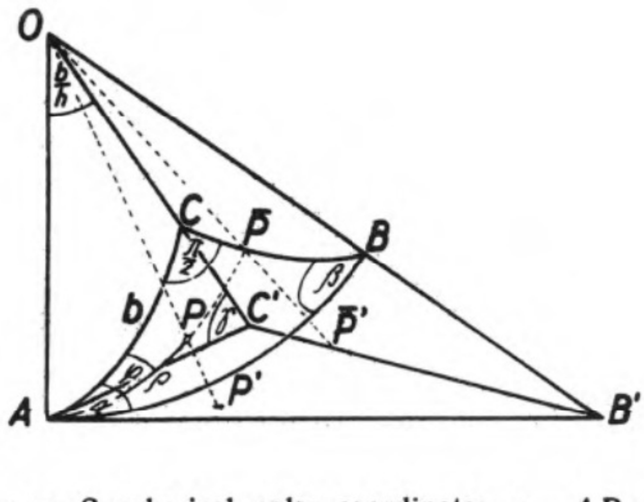
[3]) Hereafter we shall denote a point set and its measure (surface area, volume) by the same symbol; $\sqrt{\pi S}$ equals the length of a greatest circle of S.

spherical triangle ABC of angles $A=\alpha=\pi f/2e$, $B=\beta=\pi v/2e$, $C=\pi/2$. Equality holds only if N is regular and $P_1,\ldots,P_f$ are the centres of the faces $S_1,\ldots,S_f$.

Let p be a plane touching the sphere S of centre O at the point A and let P' be the radial projection of P upon p. Consider the volume G of the sphere of radius OP' as a function $G=G(AP)$ of the spherical distance AP. Defining $g(AP)=\frac{1}{S}G(AP)$, the integral $\int\limits_D g(AP)dS$ extended over a domain D lying on a hemi-sphere of centre A yields the volume of the cone of apex O and base plane p which intersects S in D. Therefore, if in the theorem mentioned above, N is the central projection of the edges of a convex polyhedron V containing S, while $OP_1,\ldots,OP_f$ are the normals of the faces and g is the function considered just now, then

$$V\geqq\sum_{i=1}^{f}\int\limits_{S_i}g(P_iP)dS,$$

with equality only if V is circumscribed about S. On the other hand $\int\limits_{\Delta}g(AP)dS$ equals the volume of the quadrirectangular tetrahedron $OAB'C'$, where $AB'C'$ is the radial projection of the right spherical triangle ABC upon the plane p. Thus, in order to obtain the desired estimating formula for V in terms of S, f, e, v and the space-constant, we have only to compute the last integral.



Introducing on S spherical polar coordinates $\varrho=AP$, $\varphi=\sphericalangle CAP$ and putting $S=4\pi h^2$, the area element becomes

$$dS=h\sin\frac{\varrho}{h}\,d\varrho d\varphi.$$

Hence the volume T of the tetrahedron $OAB'C'$ is given by

$$T=\frac{1}{S}\int_{\Delta} G\,dS=\frac{1}{4\pi h}\int_0^{\alpha}\int_0^{\bar\varrho} G\sin\frac{\varrho}{h}\,d\varrho\,d\varphi,$$

where $\bar\varrho=A\bar P$, denoting by $\bar P$ the point of intersection of the great circle AP and the side CB. We have

$$\cot\frac{\bar\varrho}{h}=\cot\frac{b}{h}\cos\varphi,$$

where the side $b=AC$ is given by

$$\cos\frac{b}{h}=\frac{\cos\beta}{\sin\alpha}.$$

We have now to express the volume G of the sphere of radius $R=AP'$ as a function of ϱ. Denoting the curvature of the space by c,

$$G=\frac{\pi}{\sqrt{c^3}}(2\sqrt{c}R-\sin 2\sqrt{c}R)$$

and

$$\cot\sqrt{c}R=\cot\sqrt{c}r\cdot\cos\frac{\varrho}{h},$$

where $r=OA$ denotes the radius of S[4]).

Introducing in the integral

$$I=\int_0^{\bar\varrho} G\sin\frac{\varrho}{h}\,d\varrho$$

the new variable R, we obtain

$$I=\frac{\pi^2 h}{c\cot\sqrt{c}r}\int_r^{\bar R}\frac{2\sqrt{c}R-\sin 2\sqrt{c}R}{\sin^2\sqrt{c}R}\,dR,$$

where $\bar R=O\bar P'$, $\bar P'$ being the intersection of $O\bar P$ and $B'C'$. Since $\bar R$ is given by

$$\cot\sqrt{c}\bar R=\cot\sqrt{c}r\cdot\cos\frac{\bar\varrho}{h},$$

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[4]) The above formulae are equivalent to $G=\frac{\pi}{\sqrt{-c^3}}(\sinh 2\sqrt{-c}R-2\sqrt{-c}R)$ and $\coth\sqrt{-c}R=\coth\sqrt{-c}r\cos\frac{\varrho}{h}$.

we find in virtue of the relation

$$2(x\cot x)' = 2\left(\cot x - \frac{x}{\sin x}\right) = \frac{\sin 2x - 2x}{\sin^2 x},$$

$$I = \frac{2\pi^2 h}{\sqrt{c^3}\cot\sqrt{c}r}\left[\sqrt{c}R\cot\sqrt{c}R\right]_{\bar{R}}^{r} = \frac{2\pi h}{c}(r - \bar{R}\cos\sqrt{c}\,\bar{\varrho}).$$

Expressing $\bar{R}$ and $\bar{\varrho}$ in terms of φ and applying the notation

$$k = \sec\frac{b}{h} = \frac{\sin\alpha}{\cos\beta},$$

we get

$$I = \frac{2\pi h}{\sqrt{c^3}}\left\{\sqrt{c}r - \frac{\cos\varphi}{\sqrt{k^2-\sin^2\varphi}}\arctan\left(\tan\sqrt{c}r\frac{\sqrt{k^2-\sin^2\varphi}}{\cos\varphi}\right)\right\}.$$

Hence, finally[5]),

$$T = \frac{1}{2\sqrt{c^3}}\int_0^\alpha\left\{\sqrt{c}r - \frac{\cos\varphi}{\sqrt{k^2-\sin^2\varphi}}\arctan\left(\tan\sqrt{c}r\frac{\sqrt{k^2-\sin^2\varphi}}{\cos\varphi}\right)\right\}d\varphi.$$

We recapitulate our result in the following theorem[6]):

If in a three-dimensional space of constant curvature $c \neq 0$ a convex polyhedron having f faces, e edges and v vertices contains a sphere of radius r, then its volume V satisfies the inequality

$$V \geqq \frac{2e}{\sqrt{c^3}}\int_0^{\pi f/2e}\left\{\sqrt{c}r - \frac{\cos\varphi}{\sqrt{k^2-\sin^2\varphi}}\arctan\left(\tan\sqrt{c}r\frac{\sqrt{k^2-\sin^2\varphi}}{\cos\varphi}\right)\right\}d\varphi,$$

where

$$k = \frac{\sin\frac{\pi f}{2e}}{\cos\frac{\pi v}{2e}}.$$

Equality holds only for a regular polyhedron circumscribed about the sphere.

Making use of the relation

$$\lim_{x\to 0}\frac{x - \lambda\arctan\frac{1}{\lambda}\tan x}{x^3} = \frac{1-\lambda^2}{3\lambda^2}$$

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[5]) E. g. in case $c = -1$ we have

$$2T = \int_0^\alpha \frac{\cos\varphi}{\sqrt{k^2-\sin^2\varphi}}\operatorname{ar\,tanh}\left(\tanh r\frac{\sqrt{k^2-\sin^2\varphi}}{\cos\varphi}\right)d\varphi - \alpha r.$$

[6]) This theorem can be extended by suitable definitions to star polyhedra. Cf. [4].

we get in the limiting case $c \to 0$

$$V \geqq \frac{2er^3}{3} \int_0^{\pi f/2e} \frac{k^2-1}{\cos^2\varphi} d\varphi = \frac{2er^3}{3}(k^2-1)\tan\frac{\pi f}{2e}.$$

This is just the inequality for Euclidean polyhedra mentioned in the introduction.

Let us still remark that the edge $r = OA$ of the tetrahedron $T = OAB'C'$ can be expressed in terms of α, β and the third non-right dihedral angle γ of T, i. e. the angle at $B'C'$. In fact, in the right triangle OAC' the angle at C' is just γ, whilst the angle at O equals b/h. Consequentely

$$\cos\sqrt{c}\,r = \frac{\cos\gamma}{\sin\frac{b}{h}} = \frac{\sin\alpha\cos\gamma}{\sqrt{\sin^2\alpha - \cos^2\beta}}.$$

Eliminating r by means of this relation from the above formula for T, we obtain the volume T in terms of α, β and γ, i. e. an explicit formula for the functions of SCHLÄFLI and LOBATSCHEFSKY. This formula can be used for numerical computation, though in the applications to the theory of regular polytopes and honeycombs it is largely superseded by the series given by COXETER [1].

As an example we compute the volume of the characteristic tetrahedron (see [2], p. 284) of the regular star-polytope $\{3, 3, 5/2\}$, which is a quadrirectangular spherical tetrahedron ($c=1$) of angles $\alpha = \pi/3, \beta = \pi/3, \gamma = 2\pi/5$. We have $k = \sqrt{3}$ and $\cos r = \sqrt{\frac{3}{2}}\,\sin 18°$, whence

$$2T = \frac{\pi}{3} r - \int_0^{\pi/3} \frac{\cos\varphi}{\sqrt{3-\sin^2\varphi}} \operatorname{arc\,tan}\left(\tan r \frac{\sqrt{3-\sin^2\varphi}}{\cos\varphi}\right) d\varphi.$$

Substituting $x = \sin\varphi$ and writeing for abbreviation

$$F(x) = \frac{\operatorname{arc\,tan}\left(\tan r \cdot \sqrt{\frac{3-x^2}{1-x^2}}\right)}{\sqrt{3-x^2}}$$

we obtain

$$2T = \frac{\pi}{3} r - \int_0^{\sqrt{3}/2} F(x)\,dx.$$

Making use of SIMSON's formula

$$2T \sim \frac{\pi}{3} r - \frac{\sqrt{3}}{12}\left\{F(0) + 4F\left(\frac{\sqrt{3}}{4}\right) + F\left(\frac{\sqrt{3}}{2}\right)\right\} = 0{,}52246\cdots.$$

The honeycomb $\{3, 3, 5/2\}$ is made up of 14400 such tetrahedra, on account of which the density[7]) d of it is given by

$$d = \frac{14400\,T}{2\pi^2} \sim 190,6.$$

The exact value of d is 191.

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[7]) d is the number of times the honeycomb will cover the (spherical) space, i. e the surface of a 4-dimensional unit sphere.