

On a problem in set theory.

To the memory of Professor Tibor Szele.

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Let E be a given non countable set of power m and suppose that there exists a relation R between the elements of E such that for any $x \in E$, the power of the set $H(x)$ of elements $y \in E$ ($y \neq x$) for which xRy holds, is smaller than a given cardinal number n which is smaller than m . Two distinct elements x and y of E are called *independent* if neither xRy nor yRx holds. We say that a subset of E is a *free set* if any two points of this subset are independent. Let $Z(x)$ denote, for every $x \in E$, the set of $y \in E$ ($y \neq x$) for which yRx holds; further, let $Z[F]$ denote, for every $F \subseteq E$ the set $\bigcup_{x \in F} Z(x)$.

The following problem is due to RUZIEWICZ [1]:

Does there always exist a free subset of power m of E ?

The answer to this problem is affirmative if $n = \aleph_0$ and m is either of the form 2^q or of the form $\aleph_{q+1}$ [2], [3], and also if m is a regular cardinal number or if m is a countable sum of cardinals smaller than m [4], [5], finally, in the general case, assuming the generalized continuum hypothesis [6].

We prove the following

Theorem. *If m is a singular cardinal number and if for every $x \in E$ the power of the set $Z(x) = \{y \in E : yRx\}$ is smaller than m , then the answer to the problem is affirmative.*

PROOF.¹⁾ Let φ be the initial number of m and φ^* the smallest ordinal number ψ for which φ is confinal to ψ . There exist regular cardinal numbers $m_1, m_2, \dots, m_\xi, \dots$ ($\xi < \varphi^*$) such that $\max\{n, \bar{\varphi}^*\} < m_\xi < m$ ($\xi < \varphi^*$), $m_\alpha > m_\beta$ for $\alpha > \beta$ and

$$m = \sum_{\xi < \varphi^*} m_\xi.$$

¹⁾ The proof is similar to the proof of a theorem of DUSHNIK, MILLER and ERDŐS. ([7] p. 606, Theorem 5.22.)

Define the free sets N_η ($\eta < \varphi^*$) by transfinite induction as follows. Let M_1 be an arbitrary subset of power m_1 of E . Since m_1 is regular and $n < m_1$, by the result of S. PICCARD [4], M_1 contains a free subset G_1 of power m_1 . Let $G_{1\xi}$ be the set of elements x of G_1 for which $\overline{Z(x)} \leq m_\xi$. It is obvious that

$$G_1 = \bigcup_{\xi < \varphi^*} G_{1\xi}.$$

Since m_1 is regular and $\bar{\varphi}^* < m_1$, there exists therefore an ordinal number $\xi_1 < \varphi^*$ such that the power of $G_{1\xi_1}$ is m_1 . It is obvious that the power of the set

$$Z[G_{1\xi_1}] = \bigcup_{x \in G_{1\xi_1}} Z(x)$$

is not greater than $m_1 \cdot m_{\xi_1}$. Put

$$N_1 = G_{1\xi_1}.$$

Let α be a given ordinal number $1 < \alpha < \varphi^*$ and suppose that the free set N_ξ is defined for every $1 \leq \xi < \alpha$ such that $\overline{N_\xi} = m_\xi$ and $\overline{Z[N_\xi]} < m$. Since $\alpha < \varphi^*$, we have that the power of the set

$$E_\alpha = E - \left(\bigcup_{\xi < \alpha} (N_\xi \cup Z[N_\xi]) \right),$$

is m . Let M_α be an arbitrary subset of power m_α of E_α . Let now N_α be defined starting from M_α in the same way as N_1 is defined starting from the set M_1 .

Put

$$B_\xi = N_\xi - \bigcup_{\gamma < \xi} \left(\bigcup_{x \in N_\gamma} H(x) \right) \quad (\xi < \varphi^*).$$

By the hypothesis $\overline{H(x)} < n$ and $\max\{n, \bar{\varphi}^*\} < m_\eta$ for every $\eta < \varphi^*$. It follows that

$$\overline{\bigcup_{x \in N_\gamma} H(x)} \leq n \cdot m_\gamma = m_\gamma.$$

Since $m_\gamma < m_\xi$ ($\gamma < \xi$) and $\bar{\xi} < \bar{\varphi}^* < m_\xi$, we have by the regularity of m_ξ that

$$\overline{\bigcup_{\gamma < \xi} \left(\bigcup_{x \in N_\gamma} H(x) \right)} < m_\xi.$$

Consequently $\overline{B_\xi} = m_\xi$. Let

$$B = \bigcup_{\xi < \varphi^*} B_\xi.$$

It is obvious that $\overline{B} = m$.

Now we prove that B is a free set. Indeed, let y and z be two distinct elements of B . If $y \in B_\alpha$ and $z \in B_\alpha$ for some $\alpha < \varphi^*$, then it follows, by the definition of N_α and the relation $B_\alpha \subseteq N_\alpha$, that y and z are independent. Let now $y \in B_\alpha$ and $z \in B_\sigma$ ($\alpha \neq \sigma$). Obviously we may suppose that $\alpha < \sigma$. It follows from

$$B_\sigma \subseteq E_\sigma = E - \bigcup_{\xi < \sigma} (N_\xi \cup Z[N_\xi])$$

that z non Ry . On the other hand, we have by the equality

$$B_\sigma = N_\sigma - \bigcup_{\xi < \sigma} \left(\bigcup_{x \in N_\xi} H(x) \right)$$

that y non Rz . Thus the set B is free, and so our theorem is proved.

Bibliography.

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