

Remark on the zeros of characteristic equations.

To the memory of my late friend, Tibor Szele.

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1. This paper contains some remarks concerning an important question of the classical algebra. This refers to the question, what a strip

$$(1.1) \quad \beta_1 \leq \Re z \leq \beta_2$$

can be given for the $\lambda'_1, \lambda'_2, \dots, \lambda'_n$ zeros of the characteristic equation

$$(1.2) \quad \Phi(\lambda) \equiv \begin{vmatrix} b_{11} - \lambda & \dots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nn} - \lambda \end{vmatrix} = 0$$

when we know that all $\lambda_1, \lambda_2, \dots, \lambda_n$ zeros of the characteristic equation

$$(1.3) \quad \psi(\lambda) \equiv \begin{vmatrix} a_{11} - \lambda & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} - \lambda \end{vmatrix} = 0$$

lie in the strip

$$(1.4) \quad \alpha_1 \leq \Re z \leq \alpha_2$$

and we know an upper estimation of

$$(1.5) \quad F \equiv \sum_{j=1}^n \sum_{k=1}^n |b_{jk} - a_{jk}|^2.$$

We assert that a lower estimation of β_1 and an upper estimation of β_2 can be given in terms of n, F and the a_{jk} numbers, at least, when F is "small enough". Obviously it is enough to deal with the lower estimation and without loss of generality we may suppose

$$(1.6) \quad \alpha_1 = 1.$$

Let us denote

$$(1.7) \quad \max_{j, k} |a_{jk}| = A$$

and let finally Δ be so large that

$$(1.8) \quad e^{2\Delta} \geq 3n \{2e(\Delta + 1)\}^{2n}$$

Then we assert the

Theorem. If

$$(1.9) \quad \sum_{j=1}^n \sum_{k=1}^n |b_{jk} - a_{jk}|^2 \leq \left(\frac{e^{-nA(\Delta^2 + \Delta)}}{2n(\Delta^2 + \Delta)} \right)^2,$$

then we have

$$\beta_1 \geq 1 - \frac{2}{\Delta + 1}.$$

2. This theorem clearly belongs to the classical algebra; curiously enough our proof will be performed via differential-equations. A direct algebraical proof based on matrix-calculus would be of interest; such a proof is easy when the (a_{jk}) and (b_{jk}) matrices are real and symmetrical. In this case the restriction (1.9) can be considerably lightened; it is very probable that the same is true in the general case.

The proof is based on the following theorem on differential-equations.¹⁾

Let the functions $w_j(z_1, z_2, \dots, z_n, t)$ for $j = 1, 2, \dots, n$ and $z_\nu = \zeta_\nu + i\eta_\nu$ (ζ_ν, η_ν reell) continuous in the halfplane $t \geq 0$ of the $(2n+1)$ -dimensional $(\zeta_1, \eta_1, \dots, \zeta_n, \eta_n, t)$ -space and consider the system

$$(2.1) \quad \frac{dY_j(t)}{dt} = \sum_{k=1}^n c_{jk} Y_k(t) + w_j(Y_1, \dots, Y_n, t) \quad j = 1, 2, \dots, n$$

with complex constants c_{jk} ; let

$$\max_{j, k} |c_{jk}| = C$$

and suppose all x_j zeros of the equation

$$(2.2) \quad G(x) \equiv \begin{vmatrix} c_{11} - x & \dots & c_{1n} \\ \vdots & & \vdots \\ c_{n1} & \dots & c_{nn} - x \end{vmatrix} = 0$$

are in the half-plane $\Re x \geq 1$. If the positive ω satisfies the inequality

$$(2.3) \quad e^{2\omega} \geq 3n \{2e(\omega + 1)\}^{2n}$$

and for the w_j -functions we have for $t \geq 0$

$$(2.4) \quad \frac{\sum_{j=1}^n |w_j(z_1, z_2, \dots, z_n, t)|^2}{\sum_{j=1}^n |z_j|^2} \leq \left\{ \frac{1}{2n(\omega^2 + \omega)} \cdot \frac{e^{-nC(\omega^2 + \omega)}}{(2e(\omega + 1))^n} \right\}^2,$$

then for each $(Y_1, Y_2, \dots, Y_n)$ solution of the system (2.1) and in each interval I in $t \geq 0$ of length $(\omega^2 + \omega)$ we have a t^* with

$$(2.5) \quad \sum_{j=1}^n |X_j(t^*)|^2 \geq e^{2t^* \left(1 - \frac{2}{\omega + 1}\right)} \sum_{j=1}^n |X_j(0)|^2.$$

¹⁾ To be published in *Acta Math. Acad. Sci. Hungar.* The main tool of the proof is the Theorem VIII. of my book "Eine neue Methode in der Analysis und deren Anwendungen", Budapest, Akad. Kiadó, 1953.

One of the main-points of this theorem is the density of these t^* -values. For our present purposes however suffices the weaker statement that there are arbitrary big t^* -values with (2.5). As easy to see from the proofs of the quoted Acta-paper, this can be assured requiring instead of (2.4) only the weaker inequality

$$(2.6) \quad \frac{\sum_{j=1}^n |w_j(z_1, \dots, z_n, t)|^2}{\sum_{j=1}^n |z_j|^2} \leq \left(\frac{e^{-nC(\omega^2 + \omega)}}{2n(\omega^2 + \omega)} \right)^2.$$

We omit a detailed repetition of the proof; the result we shall quote as the reduced Theorem.

3. In order to deduce our theorem from (2.6) we apply it with

$$(3.1) \quad c_{jk} = a_{jk} \quad 1 \leq j, k \leq n$$

and

$$(3.2) \quad w_j(z_1, \dots, z_n, t) \equiv \sum_{k=1}^n (b_{jk} - a_{jk}) z_k$$

independently of t . Then we have

$$C \equiv A, \quad G(\lambda) \equiv \psi(\lambda)$$

and the condition

$$\Re x_j \geq 1 \quad j = 1, \dots, n$$

is owing to (1.6) fulfilled. We may choose owing to (1.8)

$$(3.3) \quad \omega = A.$$

The condition (2.6) is by the choice (3.2) fulfilled, since from the inequality of Cauchy—Buniakowski we have

$$\begin{aligned} \frac{\sum_{j=1}^n |w_j(z_1, \dots, z_n, t)|^2}{\sum_{j=1}^n |z_j|^2} &= \frac{\sum_{j=1}^n \left| \sum_{k=1}^n (b_{jk} - a_{jk}) z_k \right|^2}{\sum_{j=1}^n |z_j|^2} \leq \\ &\equiv \sum_{j=1}^n \sum_{k=1}^n |b_{jk} - a_{jk}|^2 \leq \left(\frac{e^{-nA(A^2 + A)}}{2n(A^2 + A)} \right)^2 = \left(\frac{e^{-nC(\omega^2 + \omega)}}{2n(\omega^2 + \omega)} \right)^2 \end{aligned}$$

owing to (1.9) and (3.3). The continuity-requirement of $w_j(z_1, \dots, z_n, t)$ are by the choice (3.2) evidently fulfilled and thus the reduced theorem can be applied to each function-system

$$Y_1(t + t_0), \dots, Y_n(t + t_0)$$

where $(Y_1, \dots, Y_n)$ denotes an arbitrary nonvanishing solution of the system

$$(3.4) \quad \frac{dY_j(t)}{dt} = \sum_{k=1}^n b_{jk} Y_k(t) \\ j = 1, 2, \dots, n$$

where t_0 is an arbitrary positive number. If t_0 is chosen so that

$$\sum_{j=1}^n |Y_j(t_0)|^2 > 0,$$

the reduced theorem gives the existence of a sequence

$$(3.5) \quad (t_0 <) t_1 < t_2 < \dots \rightarrow +\infty$$

with

$$\sum_{j=1}^n |Y_j(t_\nu + t_0)|^2 \geq e^{2\left(1 - \frac{2}{d+1}\right)t_\nu} \sum_{j=1}^n |Y_j(t_0)|^2$$

i. e.

$$(3.6) \quad \overline{\lim}_{t \rightarrow +\infty} \frac{1}{t} \log \left(\sum_{j=1}^n |Y_j(t)|^2 \right) \geq 2 \left(1 - \frac{2}{d+1} \right).$$

On the other hand we consider a λ'_1 -zero of $\Phi(\lambda) = 0$ with the minimal real part. Owing to a part of a theorem of POINCARÉ—PERRON,²⁾ we have an integral $(Z_1, Z_2, \dots, Z_n)$ of (3.4) such that

$$(3.7) \quad \overline{\lim}_{t \rightarrow +\infty} \frac{1}{t} \log \left(\sum_{j=1}^n |Z_j(t)| \right) = \Re \lambda'_1.$$

Let $t > 0$ be now arbitrary and fixed and let j_0 be an index with

$$|Z_{j_0}(t)|^2 \geq \frac{1}{n} \sum_{j=1}^n |Z_j(t)|^2.$$

Then

$$\begin{aligned} \sqrt[n]{\sum_{j=1}^n |Z_j(t)|^2}^{1/2} &\geq \sum_{j=1}^n |Z_j(t)| \geq |Z_{j_0}(t)| \geq \\ &\geq \frac{1}{\sqrt{n}} \left(\sum_{j=1}^n |Z_j(t)|^2 \right)^{1/2}, \end{aligned}$$

i. e. from (3.7)

$$\overline{\lim}_{t \rightarrow +\infty} \frac{1}{t} \log \left(\sum_{j=1}^n |Z_j(t)|^2 \right) = 2 \Re \lambda'_1.$$

Comparing this with (3.6) we get

$$\Re \lambda'_1 \geq 1 - \frac{2}{d+1}$$

indeed.

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²⁾ Cf. PERRON: Über lineare Differentialgleichungen bei denen die unabhängige Variable reel ist, *J. Reine Angew. Math.* **142** (1932), 254–270. The special case (3.4) could have been easily treated directly too.