

Complementation of abelian normal subgroups.

In memoriam Tibor Szele.

GRAHAM HIGMAN in Oxford.

If N is a normal subgroup of the finite group G , then a subgroup C of G is called a complement of N if $NC = G$ and $N \cap C = 1$. Our object is to give a sufficient condition for the existence and conjugacy of complements in case N is abelian. By theorems of GASCHÜTZ [1] and D. G. HIGMAN [2] it is enough to consider the case of an abelian p -group N ; and the result in this case is:

Theorem 1. *Let N be an abelian normal p -subgroup of the finite group G . Suppose that G has a p' -subgroup X such that NX is normal in G , but N_0X is not, if N_0 is any normal subgroup of G properly contained in N . Then N is complemented in G , and any two complements of N are conjugate.*

(A p' -subgroup is one whose order is prime to p .)

We begin the proof with a well-known lemma from representation theory.

Lemma. *Suppose that the abelian p -group N admits the p' -group Ω as a group of operators. Then N is a direct product $S \times T$, where S is the subgroup of elements of N stable under Ω , and T admits any group Γ of operators on N which contains Ω as a normal subgroup.*

If n is the order of Ω , then n is prime to p , and so the endomorphism ring of N contains in its centre an inverse n^{-1} of the endomorphism $g \rightarrow g^n$. Thus the endomorphism ring contains the idempotent $\sigma = n^{-1} \sum \eta(g)$, where summation is over all elements of Ω , and $\eta(g)$ is the endomorphism of N induced by the element g of Ω . Thus $N = S \times T$, with $S = N^\sigma$, $T = N^{1-\sigma}$. Evidently S is the subgroup of elements of N stable under Ω ; and if t belongs to Γ , transformation by t permutes the elements of Ω , so that $\sigma t = t\sigma$, and

$$T^t = N^{(1-\sigma)t} = N^{t(1-\sigma)} = N^{1-\sigma} = T.$$

Now let G, N, X be as in the theorem, and let C be the normaliser of X in G . Let g be any element of G , and put $Y = gXg^{-1}$. Because NX is normal in G , Y is contained in NX ; and because its order is equal to that of X , and prime to that of N , it is a complement of N in NX . But N is abelian and of order prime to its index in NX , and we may therefore apply a theorem of SCHUR (cf. ZASSENHAUS [4] p. 132) to deduce that Y is conjugate to X in NX . Thus for some n in N and x in X , $gXg^{-1} = Y = nxXx^{-1}n^{-1} = nXn^{-1}$, so that $n^{-1}g$ belongs to C and g to NC . That is $NC = G$.

We now apply the Lemma, taking Γ to be G/N and Ω to be NX/N , both groups acting on N by transformation. The conclusion is that $N = S \times T$, where T is normal in G , and S is the centraliser of X in N . We observe that in fact $S = N \cap C$. That $S \subset N \cap C$ is obvious. Conversely, both $N \cap C$ and X are normal in C , and they have coprime orders. Thus they generate their direct product, and $N \cap C$ belongs to the centraliser of X . Hence TX is contained in the centraliser of S , and is therefore normal in $G = STC$. By the hypothesis of the theorem, this implies that $T = N$, so that $S = 1$. That is $N \cap C = 1$, and C is a complement of N .

Lastly, let D be another complement of N . Then $Y = NX \cap D$ is a complement of N in NX , and so, by an argument used already, is conjugate to X , say $Y = tXt^{-1}$. D is a subgroup of the normaliser of Y , and so $t^{-1}Dt$ is a subgroup of the normaliser of X , that is, of C . But $t^{-1}Dt$ is a complement of N , since D is, and so must be the whole of C . This proves the theorem.

We give next the corresponding condition when N is not necessarily a p -group.

Theorem 1a. *Let N be an abelian normal subgroup of the finite group G . Suppose that, for each prime p , there is a p' -subgroup X of G such that NX is normal in G , but not N_0X for any normal subgroup N_0 of G , properly contained in N and of p -power index in N . Then N is complemented in G , and any two complements of N are conjugate.*

The condition is, of course, trivially satisfied, by $X = 1$, for any prime which does not divide the order of N , since there are then no subgroups N_0 . If the condition is satisfied, and $N = P \times Q$, where P is a p -group and Q a p' -group, then it permits the application of Theorem 1 to show that N/Q is complemented in G/Q , and that any two complements of G/Q are conjugate. This being true for all primes p , the theorem follows from the results of GASCHÜTZ [1] and D. G. HIGMAN [2].

Next, suppose that G and N satisfy the conditions of Theorem 1, for some subgroup X , and let M/N be the greatest normal p' -subgroup of G/N . By the theorem of SCHUR, N has a complement Y in M , which we may suppose contains X . Then the conditions of Theorem 1 still hold if we replace

X by Y . For first, $NY=M$ is normal in G by assumption. But secondly, if, for any normal subgroup N_0 of G properly contained in N , N_0Y is normal, so is its intersection N_0X with NX . If H is any group containing both G and N as normal subgroups, NY/N , as a characteristic subgroup of G/N is normal in H/N , and so NY is normal in H . But if N_0 is a normal subgroup of H properly contained in N , N_0Y is not normal in H , since it is not even normal in G . Thus N is complemented in H , and two complements of N in H are conjugate. A similar argument applies to Theorem 1a, and we may therefore state:

Theorem 2. *If G and N satisfy the hypotheses either of Theorem 1 or of Theorem 1a, then N is complemented in any group H which contains both G and N as normal subgroups, and any two complements of N in H are conjugate.*

As examples of these theorems we consider the following construction. Let E be an equivalence relation defined on the rational primes. Call a group G *E-decomposable* if it is a direct product $G_1 \times G_2 \times \dots \times G_r$, where two primes dividing the order of the same G_i are equivalent under E . Thus if E is the relation of equality, *E-decomposable* is the same as nilpotent. If G is a finite group we denote by $K(G)$ the intersection of all normal subgroups of G whose factor groups are *E-decomposable*. $K(G)$ is evidently a characteristic subgroup of G ; and since direct products and subgroups of *E-decomposable* groups are again *E-decomposable*, $G/K(G)$ is *E-decomposable*, but G/L is not if L is a normal subgroup of G properly contained in $K(G)$.

We shall show that if $N=K(G)$ is abelian, the conditions of Theorem 1a are satisfied. If p is a prime, we can write $N=P \times Q$, where P is a p -group and Q is a p' -group, because N is abelian; and we can write $G/N=A/N \times B/N$, where the order of A/N is divisible only by the primes equivalent to p , and the order of B/N only by the remaining primes, because G/N is *E-decomposable*. P is a characteristic subgroup of N and so a normal subgroup of G , and its order is prime to its index in B . By the theorem of SCHUR, P has a complement X in B . Then X is a p' -group, and $B=PX=NX$ is normal. Suppose, if possible, that N_0 is a normal subgroup of G , properly contained and of p -power index in N , such that N_0X is normal. Then N_0 is of p -power index in N and of prime-to- p index in N_0X , and so is the intersection of these two groups. Thus $N_0X/N_0 = N_0X/N \cap N_0X \cong NX/N = B/N$, which is an *E-decomposable* group whose order is divisible by no prime equivalent to p . But the order of A/N_0 is divisible only by primes equivalent to p . Hence these two normal subgroups of G/N_0 generate their direct product, and this product is *E-decomposable*. Since $G=AB=ANX=AX$, this product is the whole of G/N_0 , which is not *E-decomposable*. Thus our assumption that such a group as N_0 exists

leads to a contradiction, which shows that N satisfies the condition of Theorem 1a.

By Theorem 2, $K(G)$, if abelian, is complemented in any group containing G as a normal subgroup. In particular we have:

Theorem 3. *If $K^r(G)$ is abelian for any integer r it is complemented, and any two complements are conjugate.*

The special case when $r=1$ and E is equality was proved by SCHENKMAN [3] by quite different methods.

Bibliography.

- [1] W. GASCHÜTZ, Zur Erweiterungstheorie der endlichen Gruppen, *J. Reine Angew. Math.*, **190** (1952), 93—107.
- [2] D. G. HIGMAN, Remarks on splitting extensions, *Pacific J. Math.* **4** (1954), 545—555.
- [3] E. SCHENKMAN, The splitting of certain solvable groups, *Proc. Amer. Math. Soc.* **6** (1955) 286—290
- [4] H. ZASSENHAUS, The theory of groups, *New York*, 1949.

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