

A note on regular rings.

To the memory of my beloved teacher Professor Tibor Szele.

By LÁSZLÓ KOVÁCS in Debrecen.

It is well known, that in a regular¹⁾ ring R every left (right) ideal I is idempotent (i. e. $I^2 = I$). The question, whether or not this is a characteristic property of the regular rings, seems to be of some interest. In this little note we give three theorems related to this problem. In Theorem 1 we characterize the regular rings by a condition for one-sided ideals, which in the commutative case is equivalent to the idempotency of the ideals in R . In Theorem 2 we solve an analogous problem, the result shows that the regular rings without nonzero nilpotent elements are characterized by the idempotency of their quasi-ideals. Applying this result and a theorem of A. KERTÉSZ we get Theorem 3, which is a criterion for decomposibility of rings into a direct sum of division rings. — I am indebted to A. KERTÉSZ for his valuable help.

The concept of the quasi-ideal was introduced by O. STEINFELD in [5]. A submodule M of the ring R is said to be a quasi-ideal if $RM \cap MR \subseteq M$. Elementary facts connected with this concept: a quasi-ideal is a subring, but not every subring is a quasi-ideal; the intersection of one-sided ideals is always a quasi-ideal; in the presence of a one-sided unity every quasi-ideal is intersection of one-sided ideals, etc. These and further results on quasi-ideals are to be found in [6] and [7].

In an arbitrary ring R

$$(1) \quad JL \subseteq J \cap L$$

holds for any right-ideal J and any left-ideal L of R . Our first theorem characterizes the rings in which equality holds in (1) in every case. Namely, we prove the following

Theorem 1. *An arbitrary ring R is regular if and only if*

$$(2) \quad JL = J \cap L$$

holds for every right-ideal J and left-ideal L of R .

¹⁾ In the sense of J. VON NEUMANN [4]. — Numbers in brackets refer to the bibliography at the end of this note.

First we assume (2) and show that R is regular. Let a be an arbitrary element of R . The right-ideal generated by a is the set $[an+ar]$ of all elements of the form $an+ar$ (n integer, $r \in R$). By (2)

$$[an+ar] = [an+ar] \cap R = [an+ar]R = aR$$

and so $a \in aR$. Analogously $a \in Ra$ and hence

$$a \in aR \cap Ra = aR^2a,$$

i. e. $a = axa$, R is regular.

The converse statement is almost trivial. Let R be a regular ring, by (1) we have only to show that any element a of $J \cap L$ is in JL . From $a = axa$, $a \in J$, $xa \in L$ we conclude $a \in JL$.

In what follows we shall give a characterization of regular rings without nonzero nilpotent elements, in terms of quasi-ideals.

Theorem 2. For an arbitrary ring R the following conditions are equivalent:

- $\alpha)$ R is a regular ring without nonzero nilpotent elements;
- $\beta)$ every quasi-ideal of R is idempotent;
- $\gamma)$ for every right-ideal J and every left-ideal L of R

$$JL = J \cap L \subseteq LJ$$

holds;

- $\delta)$ R is regular and isomorphic to a subdirect sum of division rings.²⁾

$\alpha)$ implies $\beta)$. Let M be a quasi-ideal of R and a an arbitrary element of M . Since M is a subring, $M^2 \subseteq M$ and so we have only to prove $M \subseteq M^2$, i. e. $a \in M^2$. By the regularity of R we have $a = axa$. Here xa is an idempotent and so, since R has no nonzero nilpotents, xa is in the center of R . Using also $MR^2M \subseteq MR \cap RM \subseteq M$ we get

$$a = (ax)a(xa) = (ax)(xa)a = (ax^2a)a \in MR^2M \cdot M \subseteq M^2,$$

qu. e. d.

$\beta)$ implies $\gamma)$. Let M and N denote quasi-ideals in R , then $M \cap N$ is also a quasi-ideal. By the idempotency of $M \cap N$ we have

$$M \cap N = (M \cap N)^2 \subseteq MN \cap NM.$$

On the other hand

$$MN \cap NM \subseteq MR \cap RM \subseteq M$$

analogously $MN \cap NM \subseteq N$, and so $M \cap N = MN \cap NM$.

Now let J be a right-ideal, L a left-ideal in R . Since a one-sided ideal is always a quasi-ideal, we have $J \cap L = JL \cap LJ$, but $JL \subseteq J \cap L$ and so $JL = J \cap L \subseteq LJ$.

²⁾ The equivalence of $\alpha)$ and $\delta)$ has been proved by A. FORSYTHE and N. H. MCCOY in [2]; we prove it here only for completeness' sake.

$\gamma)$ implies $\delta)$. Let R be a subdirect sum of the subdirectly irreducible rings $R_1, \dots, R_\nu, \dots$.³⁾ Since every R_ν is a homomorphic image of R , all the R_ν 's have property $\gamma)$. Thus, by Theorem 1, R and all the R_ν 's are regular. Suppose, that one of the R_ν 's has divisors of zero, i. e. for some nonzero elements a and b of R_ν we have $ab=0$. Then by $\gamma)$

$$bR_\nu \cdot R_\nu a = bR_\nu \cap R_\nu a \subseteq R_\nu a \cdot bR_\nu = 0$$

and so

$$R_\nu bR_\nu \cap R_\nu aR_\nu = R_\nu bR_\nu \cdot R_\nu aR_\nu = 0$$

where, by $a = (ax)a(xa) \in R_\nu aR_\nu$ and $b = (by)b(yb) \in R_\nu bR_\nu$, none of these ideals is 0. This contradicts to the supposition that R_ν is subdirectly irreducible, and so we have that all the R_ν 's are regular rings without divisors of zero, i. e. all the R_ν 's are division rings.

That $\delta)$ implies $\alpha)$ does not need a proof.

Corollary. *A commutative ring R is regular if and only if every ideal is idempotent in R .*

This follows immediately from Theorem 2, since in a commutative ring every quasi-ideal is an ideal, and a commutative regular ring can have no nonzero nilpotents. Our statement is also a simple corollary of Theorem 1, since (2) in the case $J=L$ implies the idempotency of the ideals, and the proof of the converse statement is analogous to that of $\beta)$ implies $\gamma)$ in Theorem 2.

As an application of Theorem 2 we have

Theorem 3. *A ring R is a direct sum⁴⁾ of division rings if and only if R satisfies the descending chain condition for principal (two-sided) ideals and every quasi-ideal of R is idempotent.⁵⁾*

The proof is based on the following (not yet published) theorem of A. KERTÉSZ:

Let φ be a property defined for simple rings. A ring R is a direct sum of simple rings with property φ if and only if R satisfies the descending chain condition for principal ideals and contains a system of its maximal ideals M_ν , the intersection of which is 0 and for which all the factor rings R/M_ν have property φ .

Since a direct sum R of division rings is a regular ring without nonzero nilpotent elements, by Theorem 2 every quasi-ideal of R is idempotent;

³⁾ According to a theorem of G. BIRKHOFF [1] such a representation always exists.

⁴⁾ By a direct sum we always mean a two-sided (ring-theoretical) discrete direct sum, but we use the term "subdirect sum" in the usual sense, i. e. for certain subrings of the complete direct sum.

⁵⁾ Another criterion is given by A. GERTSCHIKOFF [3].

moreover it is evident, that R satisfies the descending chain condition for principal ideals. Conversely, if R satisfies the conditions of Theorem 3, then by Theorem 2 it is a subdirect sum of division rings, i. e. there exists a system of maximal ideals M_ν in R , the intersection of which is 0 and for which every factor ring R/M_ν is a division ring. So we have by the above theorem of A. KERTÉSZ (in our case property φ characterizes the division-rings) that R is a direct sum of division rings.

REMARK. I am indebted to Professor L. FUCHS who has kindly directed my attention to the fact that

A ring R is a finite direct sum of division rings if and only if it has no nilpotent quasiideals and satisfies the descending chain condition for quasi-ideals.

This is an immediate consequence of the WEDDERBURN—ARTIN structure theorem, and also of GERTSCHIKOFF's theorem [3].

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