

Boundaries of domains of attraction

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Introduction

If $f(z)$ is an entire or rational function, the iterates $f_n(z)$ ($n=1, 2, \dots$) are defined inductively by

$$f_1(z)=f(z) \quad \text{and} \quad f_{n+1}(z)=f(f_n(z))$$

and are entire or rational respectively according as $f(z)$ is.

A first order fixpoint of $f(z)$ is a zero of $f(z)-z=0$. The *multiplier* of the fixpoint α is $f'(\alpha)$. A fixpoint α is said to be *attractive* or *non-attractive* according as $|f'(\alpha)|<1$ or $|f'(\alpha)|\geq 1$ respectively.

The main object of study in the global iteration theory is (see e.g. [2]) the subset $\mathfrak{F}$ of the plane where the sequence $\{f_n(z)\}$ is not normal. This definition implies that the complement $C\mathfrak{F}$ of $\mathfrak{F}$ is open and hence $\mathfrak{F}$ is closed. If f is non-bilinear then $\mathfrak{F}$ is perfect and non-empty.

In [2] is proved,

Result A. *If the set $\mathfrak{F}$ has an interior point, then it is identical with the extended plane.*

From the definition of attractive fixpoints follows

Result B. *The set of attractive fixpoints belongs to $C\mathfrak{F}$.*

Thus with an attractive fixpoint α there is an associated domain of normality of $\{f_n(z)\}$ in which it can be easily shown that $\lim_{n \rightarrow \infty} f_n(z) \rightarrow \alpha$. For such a domain we introduce the following definition.

Definition 1. The immediate domain of attraction D_α of a first order attractive fixpoint α is the maximal domain of normality of $\{f_n(z)\}$ which contains α . In D_α we have $\lim_{n \rightarrow \infty} f_n(z) = \alpha$.

FATOU [2] proved that if $f(z)$ is a rational function and α a first order attractive fixpoint of $f(z)$, then there always exists a first order fixpoint on the boundary ∂D_α of D_α . Since ∂D_α is in $\mathfrak{F}$ this fixpoint is non-attractive. Further it is shown that this non-attractive fixpoint is an accessible boundary point of D_α . A simple example is provided by $f(z)=z^2$ which has the attractive fixpoint $\alpha=0$ for which D_α is $\{|z|<1\}$, $\mathfrak{F}$ is $|z|=1$ and 1 is a non-attractive fixpoint.

When $f(z)$ is entire transcendental Fatou's result is no longer generally true. In fact an entire transcendental function may not have any non-attractive fixpoint of first order at all, provided the rate of growth of the function is not too small. Indeed WHITTINGTON [3] has proved the following theorem [3, p. 534].

Theorem C. *For every $t < 0$ there exists an entire function of order $\frac{1}{2}$ and type t for which there is no non-attractive fixpoint of first order and infinity of first order attractive fixpoints.*

In this paper we prove

Theorem D. *Let $f(z)$ be entire transcendental and α be a first order attractive fixpoint of $f(z)$ such that D_α is bounded. Then there exists a first order fixpoint on the boundary ∂D_α of D_α , which is accessible from the interior of D_α .*

While for a rational function it is clear that the immediate domain of attraction of an attractive fixpoint is bounded, an entire function may have unbounded domains of attraction. Theorem D shows that if all the first order fixpoints of an entire transcendental function are attractive the the immediate domains of attraction of each of these fixpoints must be unbounded. This is the case e.g. for Whittington's examples.

There are indeed examples of entire transcendental functions having an attractive fixpoint α for which D_α is bounded. BAKER [1] has constructed an entire function $g(z)$ for which $g(0)=0$, $g'(0)=0$, so that 0 is an attractive fixpoint and in its immediate domain of attraction $g_n(z) \rightarrow 0$. However, there are annuli $R < |z| < R'$ in which $g_n(z) \rightarrow \infty$, and so D_0 lies completely inside $|z| < R$.

Proof of theorem D

We have $f(\alpha)=\alpha$, $|f'(\alpha)| < 1$, $f(D_\alpha) \subset D_\alpha$, D_α bounded.

Take the brach or branches of $f_{-1}(z)$ such that $f_{-1}(\alpha)=\alpha$. Under analytic continuation within D_α , these branches always yield values $w=f_{-1}(z)$ lying in D_α , since $\mathfrak{F}$ and $C\mathfrak{F}$ are completely invariant in the sense that $z \in \mathfrak{F}$ implies $f(z) \in \mathfrak{F}$ and $z \in C\mathfrak{F}$ implies $f(z) \in C\mathfrak{F}$, also $f(z) \in \mathfrak{F}$ whenever $z \in \mathfrak{F}$ and $f(z) \in C\mathfrak{F}$ whenever $z \in C\mathfrak{F}$, cf. [2].

Throughout this proof let $\hat{f}_{-1}(z)$ be the inverse of the restriction of $f(z)$ to D_α .

Under the continuation of a branch of $\hat{f}_{-1}(z)$ along a path γ in D_α we never encounter any transcendental singularity, for this would imply that the values of $|\hat{f}_{-1}(z)| \rightarrow \infty$ as z traverses a part of $\gamma \subset D_\alpha$; this in turn would imply that $D_\alpha (\cong \hat{f}_{-1}(\gamma))$ is unbounded.

An algebraic singularity β of a branch of $\hat{f}_{-1}(z)$ in D_α corresponds to a zero b_1 of $f'(z)$: $f'(b_1)=0$, $f(b_1)=\beta$ and $f_{-1}(\beta)=b_1$ for the appropriate branch of $\hat{f}_{-1}$. Since $\beta \in D_\alpha$ we see that $b_1 \in D_\alpha$. Now D_α is bounded and so contains only finitely many zeros $b_1, b_2, \dots, b_n$, of $f(z)$ in D_α . Thus $\hat{f}_{-1}(z)$ is a function with a finite number n of branches in D_α . Moreover $f_m(b_j) \rightarrow \alpha$ as $m \rightarrow \infty$, for each j .

Thus the set S of singularities of all $\hat{f}_{-m}(z)$ ($=$ inverse of the restriction of $f_m(z)$ to D_α) is precisely the set of all $f_m(b_j)$ ($m=1, 2, \dots$; $j=1, 2, \dots, n$) and this

is a countable subset of D with just one limit point at α . The set S is thus closed and bounded and hence it is compact.

Thus there exists a $\xi \in D_\alpha$, $\xi \neq \alpha$, $\xi \notin S$. Then if we pick any branch $p(z)$ of $f_{-1}(\xi) = \xi_1$, we have $\xi_1 \notin S$, $\xi_1 \neq \alpha$. We can join ξ to ξ_1 in $D_\alpha - S$, say by a polygonal arc l which will have positive distance from the compact set S . We can therefore enclose l in a domain $\delta (\subset D_\alpha)$ which does not meet S (e.g. the domain δ may be all points distant less than ε from l , for all small $\varepsilon > 0$).

Now continue $p(z)$ along l to ξ_1 . p maps l onto l_1 , joining ξ_1 to ξ_2 , say. Also p extends to a single valued branch of f_{-1} in δ (by the Monodromy Theorem) and maps δ one to one onto δ_1 , another simply connected which contains l_1 within it and which does not meet S . Similarly by further continuation we get a chain of points $\xi, \xi_1, \xi_2, \dots$, and curves $l, l_1, l_2, \dots$, contained in simply connected domains $\delta, \delta_1, \delta_2, \dots$, and such that l_n joins ξ_n to ξ_{n+1} , $p(z)$ continues analytically along $l + l_1 + \dots$, and is regular in each δ_n , since no δ_n meets S .

Let p_n denote the n th continuation of p by the above process. Then p_n maps δ_n one to one onto δ_{n+1} . This is equivalent to saying that $f(z)$ maps δ_{n+1} one to one onto δ_n . The compound map $q_n = p_n \cdot p_{n-1} \dots p_1 \cdot p$ is a branch of $f_{-(n+1)}(z)$, regular in δ mapping δ onto $\delta_{n+1} \subset D_\alpha$.

Consider the branches q_n . These are regular and bounded in δ and hence normal in δ . We shall show that the limit function $\psi(z)$ of any convergent subsequence q_{n_k} is a constant on the boundary ∂D_α of D_α . For $q_{n_k}(\delta) \subset D_\alpha$. Hence $\psi(z) \in \bar{D}_\alpha$. However for any compact $A \subset D_\alpha$, $f_n(A) \rightarrow \alpha$ as $n \rightarrow \infty$ and hence for large n , $f_n(A) \cap \delta = \emptyset$, since δ has positive distance from α . Thus for large n_k , $q_{n_k}(\delta)$ does not meet A . Hence $\psi(z)$ must be on the boundary of $\bar{D}_\alpha (= \partial D_\alpha)$. But ∂D_α contains no open set since $\partial D_\alpha \subset \mathfrak{F}$ and $\mathfrak{F}$ contains no open set unless $\mathfrak{F}$ is identically equal to the extended plane (c.f. result A.) which is not the case under the assumption of our theorem. Hence $\psi(z)$ must be a constant. Suppose now $\lim_{k \rightarrow \infty} q_{n_k}(z) \rightarrow \lambda$, $z \in \delta$, i.e.

$$\lim_{k \rightarrow \infty} q_{n_k}(\delta) = \lim_{k \rightarrow \infty} \delta_{n_k+1} = \lambda.$$

Then $\delta_{n_k+2} \rightarrow \lambda$ since $\delta_{n_k+1} \cap \delta_{n_k+2} \neq \emptyset$, and $\text{diam } \delta_{n_k+2} \rightarrow 0$. But

$$f(\delta_{n_k+2}) = \delta_{n_k+1}, \text{ and hence when } k \rightarrow \infty \\ f(\lambda) = \lambda$$

Thus $\lambda \in \mathfrak{F}$ is a first order fixpoint of $f(z)$ which must lie on the boundary ∂D_α of D_α .

It remains now to show that λ is an accessible boundary point of D_α . FATOU [2] only outlined the proof, but we shall give a detailed proof here.

We define $L = \limsup \delta_n$ as the set of those points t for which there exists a sequence of integers N_q and points $z_q \in \delta_{N_q}$ such that $N_q \rightarrow \infty$, $z_q \rightarrow t$ (see e.g. [4], p. 10).

Then $L \subset \partial D_\alpha$, because each t is a limit function of a sequence of $\{q_{n_k}\}$ and we have already shown that this limit must lie on ∂D_α . Furthermore any $t \in L$ is a first order fixpoint of $f(z)$ by the arguments in the proof of the first part of the theorem. It follows from the definition that L is closed. Further L is bounded, since $L \subset \partial D_\alpha$ and D_α is bounded. Hence L is compact. We now show that L is connected.

Suppose L is not connected. Then there exist closed nonempty sets L_1 and L_2 such that $L \subset L_1 \cup L_2$, $L_1 \cap L_2 = \emptyset$ and the distance between L_1 and L_2 is

$$\varrho(L_1, L_2) = 4\eta \text{ (say) where } \eta > 0,$$

Now for some N_0 , δ_n lies in an η -neighbourhood of L for all $n > N_0$. Otherwise there exist $n_1, n_2, \dots$ tending to infinity and $\xi_{n_1} \in \delta_{n_1}, \xi_{n_2} \in \delta_{n_2}, \varrho(\xi_{n_i}, L) \equiv \eta$, and ξ_{n_i} will have a point of accumulation (since they are inside the compact set D_α) t and $t \notin L$. This is against the definition of L .

Take $l_1 \in L_1$ and $l_2 \in L_2$. Then there exists a $\delta_{n_1}, n_1 > N_0$ which meets an η -neighbourhood of l_1 and similarly there exists a $\delta_{n_2}, n_2 > n_1$ which meets an η -neighbourhood of l_2 .

Consider the chain

$$C = \delta_{n_1} \cup \delta_{n_1+1} \cup \dots \cup \delta_{n_2},$$

where δ_{n_1} meets an η -neighbourhood of l_1 and δ_{n_2} meets an η -neighbourhood of l_2 .

Now C must be connected since $\delta_n \cap \delta_{n+1} \neq \emptyset$. Also C lies in an η -neighbourhood of L i.e. of L_1 and L_2 . Given $\mu_1 \in \delta_{n_1} \subset L'_1 = \eta$ -neighbourhood of L_1 and $\mu_2 \in \delta_{n_2} \subset L'_2 = \eta$ -neighbourhood of L_2 there exists a polygon of sides less than η with vertices at $\mu_1 = \mu_1^1, \mu_1^2, \dots, \mu_1^n = \mu_2$ all lying in C . Let μ_1^r be the last μ_1 in L'_1 . Then $r \neq n$ since $\mu_2 = \mu_1^n$ lies in L'_2 . Also $\mu_1^{r+1} \in L'_2$. Now $\varrho(\mu_1^r, \mu_1^{r+1}) < \eta$

$$\text{i.e.} \quad \varrho(L_1, L_2) < \varrho(L_1, \mu_1^r) + \varrho(\mu_1^r, \mu_1^{r+1}) + \varrho(\mu_1^{r+1}, \mu_2) < 3\eta$$

which is a contradiction. We have now proved that L is connected and compact. Hence it is a continuum.

We now show that L is in fact a single point. If L is not a single point then we shall have a continuum of first order fixpoints of $f(z)$ lying on the finite part of the plane i.e. $f(z) - z = 0$ shall have infinity of solutions in a bounded area. This is impossible unless $f(z) \equiv z$. Thus L must reduce to a single point. This means that the whole sequence $q_n(z_0)$ for $z_0 \in \delta$ must tend uniformly to the single frontier point λ i.e. the chain $C = \delta_1 \cup \delta_2 \cup \dots$, which is connected tends uniformly to the frontier point λ . Hence λ is an accessible frontier (boundary) point of D_α , accessible from the interior of D_α , along the curve $l + l_1 + l_2 + \dots$.

The proof of the theorem is now complete.

The result of this paper was included in the author's Ph. D. thesis submitted to the University of London, under the supervision of Dr. I. N. BAKER. The author wishes to acknowledge his deep gratitude to Dr. Baker for his help and guidance in preparing the paper and for suggesting its subject. The author is grateful to the Govt. of Assam (India) for financial support.

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(Received April 28, 1969.)