Trigonometric identities and quadratic residues

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Abstract. In this paper, we obtain some novel identities involving trigonometric functions. Let n be any positive odd integer. We mainly show that

$$\sum_{r=0}^{n-1} \frac{1}{1 + \sin 2\pi \frac{x+r}{n} + \cos 2\pi \frac{x+r}{n}} = \frac{(-1)^{(n-1)/2} n}{1 + (-1)^{(n-1)/2} \sin 2\pi x + \cos 2\pi x}$$

for any complex number with $x + 1/2, x + (-1)^{(n-1)/2}/4 \notin \mathbb{Z}$, and

$$\sum_{j,k=0}^{n-1} \frac{1}{\sin 2\pi \frac{x+j}{n} + \sin 2\pi \frac{y+k}{n}} = \frac{(-1)^{(n-1)/2} n^2}{\sin 2\pi x + \sin 2\pi y}$$

for all complex numbers x and y with $x+y, x-y-1/2 \notin \mathbb{Z}$. We also determine the values of $\prod_{k=1}^{(p-1)/2}(1 + \tan \pi \frac{k^2}{p})$ and $\prod_{k=1}^{(p-1)/2}(1 + \cot \pi \frac{k^2}{p})$ for any odd prime p. In addition, we pose several conjectures on the values of $\prod_{k=1}^{(p-1)/2}(x - e^{2\pi i k^2/p})$ with p an odd prime and x a root of unity.

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