

A note on positive integer solutions of the equation

$$xy + yz + zx = n$$

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Abstract. Let n be a positive integer. In this note we prove that if $n > 10^{13}$, then there exist at most seven exceptional values n for which the equation $xy + yz + zx = n$ has no positive integers (x, y, z) . Moreover, under the assumption of the generalized Riemann conjecture, the above-mentioned exceptional values do not exist.

1. Introduction

Let $\mathbb{Z}, \mathbb{N}$ be the sets of integers and positive integers respectively. For a fixed positive integer n , let $S(n)$ denote the number of solutions (x, y, z) of the equation

$$(1) \quad xy + yz + zx = n, \quad x, y, z \in \mathbb{N}, \quad x \leq y \leq z.$$

In [6], KOVÁCS examined that if $n \leq 10^7$, then $S(n) > 0$ except $n = 1, 2, 4, 22, 30, 42, 58, 70, 78, 102, 130, 190, 210, 330$ and 462 . Let $E(X)$ denote the number of $n \leq X$ for which $S(n) = 0$. CAI [1] proved that $E(X) = O(X \cdot 2^{-(1-\varepsilon)(\log X)/\log \log X})$ for any $\varepsilon > 0$. Recently, HASSAN, BRINDZA and PINTÉR [2] showed that if $S(n) = 0$, then the squarefree part of n belongs to a finite set which can be effectively determined up to at most one element. While $S(n) = 0$, n is called an exceptional value. In this note we prove the following result.

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Theorem. *If $n > 10^{13}$, then there exist at most seven exceptional values n . Moreover, under the assumption of the generalized Riemann conjecture, the above-mentioned exceptional values do not exist.*

2. Preliminaries

Lemma 1. *If $n > 1$ and $2 \nmid n$ or $n > 4$ and $4 \mid n$, then $S(n) > 0$.*

PROOF. If $n > 1$ and $2 \nmid n$, then (1) has a solution $(x, y, z) = (1, 1, (n-1)/2)$. If $n > 4$ and $4 \mid n$, then (1) has a solution $(x, y, z) = (2, 2, n/4 - 1)$. The lemma is proved.

Lemma 2 ([7]). *For any positive integer t , let p_t denote the t -th odd prime. Then $p_t \geq \max(3, (t+1) \log(t+1))$.*

Lemma 3. *For any positive integer k ,*

$$p_1 p_2 \cdots p_k > \frac{5}{2} \left(\frac{k+1}{e} \right)^{k+3/2} \prod_{t=3}^{k+1} \log t.$$

PROOF. Using Lemma 2, we get

$$(2) \quad p_1 p_2 \cdots p_k > \frac{3}{2} (k+1)! (\log 3) \prod_{t=3}^{k+1} \log t.$$

By Stirling's theorem, we have

$$(3) \quad (k+1)! > \sqrt{2\pi(k+1)} \left(\frac{k+1}{e} \right)^{k+1}.$$

Substituting (3) into (2), we obtain the lemma immediately.

Lemma 4. *For any positive integer k , let $P(k) = \sqrt{8p_1 p_2 \cdots p_k} / \log 8p_1 p_2 \cdots p_k$. If $k \geq 5$, then $P(k+1) > 2P(k)$.*

PROOF. Since p_i ($i = 1, 2, \dots, k$) are all odd primes for which do not exceed p_k , every prime factor q of $8p_1 p_2 \cdots p_k - 1$ satisfies $q \geq p_{k+1}$. It implies that $8p_1 p_2 \cdots p_k - 1 \geq p_{k+1}$. Hence, if $k \geq 5$, then we have

$$\frac{P(k+1)}{P(k)} = \sqrt{p_{k+1}} \frac{\log 8p_1 p_2 \cdots p_k}{\log 8p_1 p_2 \cdots p_k + \log p_{k+1}} > \frac{1}{2} \sqrt{p_{k+1}} \geq \frac{1}{2} \sqrt{17} > 2.$$

The lemma is proved.

We now recall some basic properties on class numbers of binary quadratic forms (see [3]). For any positive integer m , let $h(m)$ and $h_0(m)$ denote the class number of binary quadratic forms and binary quadratic primitive forms with discriminant $-m$, respectively. Then we have

$$(4) \quad h(m) = \sum_{d^2|m} h_0\left(\frac{m}{d^2}\right),$$

where d^2 runs through all square divisors of m . If $m \equiv 0$ or $3 \pmod{4}$, then the negative discriminant $-m$ can be written as

$$(5) \quad -m = -fg^2,$$

where $-f$ is a fundamental discriminant, g is a positive integer. Further, we have

$$(6) \quad h_0(m) = h_0(fg^2) = h_0(f) \prod_{p|g} \left(1 - \left(\frac{-f}{p}\right) \frac{1}{p}\right)$$

and

$$(7) \quad h_0(f) = \frac{\sqrt{f}}{\pi} L(1, \chi), \quad \text{if } f > 4,$$

where p runs through all prime factors of g , $(-f/p)$ is Kronecker's symbol, χ is a real primitive character modulo f , $L(s, \chi)$ is the Dirichlet L -function associated with χ . Furthermore, let $\omega(f)$ denote the number of distinct prime factors of f , then we have

$$(8) \quad 2^{\omega(f)-1} \mid h_0(f).$$

Lemma 5 ([4, 5]). *Let χ be a real primitive character modulo q . For any positive number ε with $\varepsilon \leq 0.0723$, if $q > e^{1/\varepsilon}$, then*

$$L(1, \chi) > \min\left(\frac{1}{7.735 \log q}, \frac{2.865\varepsilon}{q^\varepsilon}\right),$$

except at most one exceptional modulo q . Moreover, under the assumption of the generalized Riemann conjecture, the exceptional modulo does not exist.

3. Proof of Theorem

Let n be an exceptional value with $n > 10^{13}$. By Lemma 1, we have $2 \parallel n$. Then n can be written in the form

$$(9) \quad n = 2\ell_1\ell_2 \cdots \ell_k u_1^{2\alpha_1} u_2^{2\alpha_2} \cdots u_r^{2\alpha_r} v_1^{2\beta_1+1} v_2^{2\beta_2+1} \cdots v_s^{2\beta_s+1},$$

where $\ell_1, \ell_2, \dots, \ell_k, u_1, u_2, \dots, u_r, v_1, v_2, \dots, v_s$ are distinct odd primes, $\alpha_1, \alpha_2, \dots, \alpha_r, \beta_1, \beta_2, \dots, \beta_s$ are positive integers. We may assume that $\ell_1 < \ell_2 < \dots < \ell_k, u_1 < u_2 < \dots < u_r$ and $v_1 < v_2 < \dots < v_s$. Notice that $3!S(n) \geq 3h(4n) - 3d(n)$ by [2, page 201], where $d(n)$ is the number of positive divisors of n . Then we have

$$(10) \quad d(n) \geq h(4n).$$

By (5) and (9), we have

$$(11) \quad -4n = -fg^2,$$

where

$$(12) \quad f = 8\ell_1\ell_2 \cdots \ell_k v_1 v_2 \cdots v_s, \quad g = u_1^{\alpha_1} u_2^{\alpha_2} \cdots u_r^{\alpha_r} v_1^{\beta_1} v_2^{\beta_2} \cdots v_s^{\beta_s}.$$

and $-f$ is a fundamental discriminant. Further, by (4), (6), (11) and (12), we get

$$(13) \quad \begin{aligned} h(4n) &= \sum_{d^2|4n} h_0\left(\frac{4n}{d^2}\right) = \sum_{d^2|g^2} h_0\left(f \frac{g^2}{d^2}\right) \\ &= \sum_{d^2|g^2} h_0(f) \left(\prod_{p|g/d} \left(1 - \left(\frac{-f}{p}\right) \frac{1}{p}\right) \right) \\ &= h_0(f) \sum_{d|g} \left(\prod_{p|g/d} \left(1 - \left(\frac{-f}{p}\right) \frac{1}{p}\right) \right). \end{aligned}$$

Since $(-f/p) \leq 1$, we deduce from (12) and (13) that

$$(14) \quad \begin{aligned} h(4n) &\geq h_0(f) \sum_{d|g} \left(\prod_{p|g/d} \left(1 - \frac{1}{p}\right) \right) \\ &= h_0(f) \left(\prod_{i=1}^r (u_i^{\alpha_i} - 1) \right) \left(\prod_{j=1}^s (v_j^{\beta_j} - 1) \right). \end{aligned}$$

On the other hand, by (9), we have

$$(15) \quad d(n) = 2^{k+1} \left(\prod_{i=1}^r (2\alpha_i + 1) \right) \left(\prod_{j=1}^s (2\beta_j + 2) \right).$$

The combination of (10), (14) and (15) yields

$$(16) \quad 2^{k+s+1} \geq h_0(f) \left(\prod_{i=1}^r \frac{u_i^{\alpha_i} - 1}{2\alpha_i + 1} \right) \left(\prod_{j=1}^s \frac{v_j^{\beta_j} - 1}{\beta_j + 1} \right).$$

We first consider the case of $g = 1$. Then we have $r = s = 0$, and by (16), we get

$$(17) \quad 2^{k+1} \geq h_0(f).$$

Using Lemma 5, by (7), we have

$$(18) \quad h_0(f) = \frac{\sqrt{f}}{\pi} L(1, \chi) > \begin{cases} \frac{\sqrt{f}}{7.735\pi \log f}, & \text{if } 4 \cdot 10^{13} < f \leq 10^{28}, \\ \frac{0.2071 f^{0.4277}}{\pi}, & \text{if } f > 10^{28}, \end{cases}$$

except at most one exceptional value. Let p_t denote the t -th odd prime. We see from (12) that $f \geq 8p_1 p_2 \cdots p_k$. Therefore, by Lemma 3, we get

$$(19) \quad f > 20 \left(\frac{k+1}{e} \right)^{k+3/2} \prod_{t=3}^{k+1} \log t.$$

Substituting (19) into (18), if $f > 10^{28}$, then from (17) and (18) we get

$$(20) \quad 654 (12.77)^k > (k+1)^{k+3/2} \prod_{t=3}^{k+1} \log t.$$

We calculate from (20) that $k \leq 10$. Since $f > 10^{28}$, by (17) and (18), we get $8 \cdot 10^3 > 2^{k+1} \pi > 0.2071 f^{0.4277} > 10^{10}$, a contradiction.

If $4 \cdot 10^{13} < f \leq 10^{28}$, then from (17) and (18) we get

$$(21) \quad 2^{k+1} > \frac{\sqrt{f}}{7.735\pi \log f}.$$

By Lemma 4, if $k \geq 11$, then from (21) we get

$$(22) \quad \begin{aligned} 10^5 \cdot 2^{k-11} &> 2^{k+1} \cdot 7.735 \pi > \frac{\sqrt{f}}{\log f} \geq \frac{\sqrt{8p_1 p_2 \cdots p_k}}{\log 8p_1 p_2 \cdots p_k} \\ &> 2^{k-11} \frac{\sqrt{8p_1 p_2 \cdots p_{11}}}{\log 8p_1 p_2 \cdots p_{11}} > 1.6 \cdot 10^5 \cdot 2^{k-11}, \end{aligned}$$

a contradiction. So we have $k \leq 10$, and by (21),

$$(23) \quad 49767 > \frac{\sqrt{f}}{\log f}.$$

Since $f > 4 \cdot 10^{13}$, (23) is impossible. Thus, there exists at most one exceptional value n such that $n > 10^{13}$ and $g = 1$. Moreover, by Lemma 5, under the assumption of the generalized Riemann conjecture, the above-mentioned exceptional value does not exist.

We next consider the case of $g > 1$. By (8) and (12), we have $h_0(f) \geq 2^{k+s}$. Hence, by (16), we obtain

$$(24) \quad 2 \geq \left(\prod_{i=1}^r \frac{u_i^{\alpha_i} - 1}{2\alpha_i + 1} \right) \left(\prod_{j=1}^s \frac{v_j^{\beta_j} - 1}{\beta_j + 1} \right).$$

Recall that $u_1, u_2, \dots, u_r, v_1, v_2, \dots, v_s$ are distinct odd primes satisfying $u_1 < u_2 < \cdots < u_r$ and $v_1 < v_2 < \cdots < v_s$. We find from (24) that

$$(25) \quad \begin{aligned} r = 1, \quad s = 0, \quad (u_1, \alpha_1) &= (3, 1), (3, 2), (5, 1), (7, 1); \\ r = 2, \quad s = 0, \quad (u_1, u_2, \alpha_1, \alpha_2) &= (3, 5, 1, 1), (3, 7, 1, 1); \\ r = 0, \quad s = 1, \quad (v_1, \beta_1) &= (3, 1), (5, 1); \\ r = 1, \quad s = 1, \quad (u_1, v_1, \alpha_1, \beta_1) &= (3, 5, 1, 1), (3, 7, 1, 1), (5, 3, 1, 1), (7, 3, 1, 1). \end{aligned}$$

From (12) and (25), we obtain

$$(26) \quad g \in \{3, 5, 7, 9, 15, 21\}.$$

Notice that

$$(27) \quad \left(\prod_{i=1}^r \frac{u_i^{\alpha_i} - 1}{2\alpha_i + 1} \right) \left(\prod_{j=1}^s \frac{v_j^{\beta_j} - 1}{\beta_j + 1} \right) > \frac{2}{3}.$$

We get from (16) and (27) that

$$(28) \quad 2^{k+s} \cdot 3 \geq h_0(f).$$

Using the same method as in the case of $g = 1$, we can prove that there exists at most one f for which (28) holds. Moreover, under the assumption of the generalized Riemann conjecture, such f does not exist. Thus, by (26), there exist at most six exceptional values n with $g > 1$. To sum up, the proof is complete.

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