

Some remarks on central idempotents in group rings

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Abstract. Let G be a group, K a field of characteristic 0, and T the set of all elements of finite order in G . In this note we give necessary and sufficient conditions under which every idempotent of KT is central in KG .

1. Introduction

Let G be a group and K a field of characteristic 0. We denote by $U(KG)$ the group of units of the group ring of G over K . Also, if X is a group, we shall denote by $T(X)$ the *torsion* of X ; i.e., the set of all elements of finite order in X . The study of group theoretical properties of $U(KG)$ has lead, on occasions, to the condition that $T = T(G)$ is a subgroup and that every idempotent of KT is central in KG . In what follows we study this condition and prove the following

Theorem 1. *Let K be a field of characteristic 0 and let T be the set of elements of finite order of a group G . Then, every idempotent of KG with support in T is central in KG if and only if the following conditions hold:*

- (i) *For every element $t \in T$ and every $x \in G$ there exists a positive integer j such that $xtx^{-1} = t^j$.*
- (ii) *If $t \in T$ is an element of order n and ζ_n is a primitive n th root of unity over K , then, for each exponent j obtained as in (i), there exists a map $\sigma \in \text{Gal}(K(\zeta_n) : K)$ such that $\sigma(\zeta_n) = \zeta_n^j$.*

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- (iii) T is either an abelian group or a Hamiltonian group such that for each element $t \in T$ of odd order k , the field $K(\zeta_k)$ contains no non-trivial solution of the equation $x^2 + y^2 + z^2 = 0$.

This problem was first studied in [1] where, due to an oversight, condition (ii) was stated in the following weaker form:

- (ii') For every non-central element t of order n , K contains no root of unity of order n .

This condition is actually not sufficient, as is shown below. The authors are grateful to Prof. E. JESPER, who spotted the oversight and suggested this counterexample.

Example. Let $G = \langle x, y \mid yxy^{-1} = x^5, x^8 = 1 \rangle$; let ζ be a primitive root of unity of order 8 and set $K = \mathbb{Q}(\zeta + \zeta^{-1})$.

Write:

$$e_1 = \frac{1}{8} \sum_{i=0}^7 (\zeta x)^i \quad \text{and} \quad e_2 = \frac{1}{8} \sum_{i=0}^7 (\zeta^{-1} x)^i.$$

Then, e_1, e_2 are orthogonal idempotents in $\mathbb{Q}(\zeta)\langle x \rangle$ and thus $e = e_1 + e_2 \in K\langle x \rangle$ is also an idempotent. Notice that the coefficient of x in e is equal to $\sqrt{2}/8$ and the coefficient of x^5 is equal to $-\sqrt{2}/8$. It follows that the coefficient of x in the element yey^{-1} is $-\sqrt{2}/8$ so $e \neq yey^{-1}$ and thus e is not central in KG . Since K is a real field, it contains no root of unity other than ± 1 and it is easily seen that conditions (i) and (iii) are also satisfied.

2. Proof of the Theorem

The fact that conditions (i) and (iii) are necessary follows as in [1]. To prove (ii), first notice that, given an element $t \in T$ we can write $K\langle t \rangle$ as a direct sum

$$K\langle t \rangle \cong K_1 \oplus \cdots \oplus K_s$$

with $K_i = (K\langle t \rangle)e_i \cong K(\zeta_i)$, where e_i , $1 \leq i \leq s$, is the set of principal idempotents of $K\langle t \rangle$, each ζ_i denotes a root of unity and at least one of them, ζ_1 say, is such that $o(\zeta_1) = o(t)$. Also notice that, in the isomorphism above, the element t corresponds to $(\zeta_1, \dots, \zeta_s)$.

Since every idempotent of KT is central in KG , it follows that conjugation by an element $x \in G$ induces an automorphism $\theta : K\langle t \rangle \rightarrow K\langle t \rangle$ which, in turn, induces automorphisms θ_i on each simple component K_i , $1 \leq i \leq s$.

Each K_i contains $\overline{K}_i = Ke_i$, which is an isomorphic copy of K and, since $xtx^{-1} = x^j$ for some positive integer j , we see that, in particular, $\theta_1 : K(\zeta_1) \rightarrow K(\zeta_1)$ fixes K and is such that $\theta_1(\zeta_1) = \zeta_1^j$ so $\theta_1 \in \text{Gal}(K(\zeta_1) : K)$, as desired.

To prove sufficiency we use the same methods as in [1] showing that they work also in the present case. Assume first that T is an abelian group and let $e \in KT$ be an idempotent. When considering $\text{supp}(e)$ we may assume that T is finite. Furthermore, since every idempotent is a sum of primitive idempotents, we may restrict ourselves to the case where e is itself primitive. We wish to show that, for each fixed element $x \in G$, we have $xex^{-1} = e$.

Write $T = \langle t_1 \rangle \times \cdots \times \langle t_s \rangle$, a direct product of cyclic groups and set $t_0 = t_1 \cdots t_s$. Then $xt_0x^{-1} = t_0^j$ for some positive integer j and thus also $xtx^{-1} = t^j$, for all $t \in T$. Notice that $o(t_0)$, the order of t_0 , is equal to the exponent of T . So if ζ is a primitive root of unity whose order is equal to $o(t_0)$, then $K(\zeta)$ is a splitting field for T . Hence, $e \in KT \subset K(\zeta)T$ is a sum of primitive idempotents of $K(\zeta)T$. Let f be one of these idempotents.

Every K -automorphism of $K(\zeta)$ extends in a natural way to an automorphism of $K(\zeta)T$. We define:

$$H = \{\phi \in \text{Gal}(K(\zeta) : K) \mid \phi(f) = f\}$$

and take $\phi_1 = I, \phi_2, \dots, \phi_r$ a transversal of H in $\text{Gal}(K(\zeta) : K)$.

Set $e^* = \phi_1(f) + \cdots + \phi_r(f)$. Exactly as in [1], it can be shown that $e^* = e$.

According to [5, Theorem 2.12], we can write f in the form:

$$f = \frac{1}{|T|} \sum_{t \in T} \chi(t^{-1})t$$

where χ is an irreducible character of T with values in $K(\zeta)$.

Then,

$$e = \frac{1}{|T|} \sum_{t \in T} \left(\sum_i \phi_i (\chi(t^{-1})) \right) t$$

so

$$xex^{-1} = \frac{1}{|T|} \sum_{t \in T} \left(\sum_i \phi_i (\chi(t^{-1})) \right) t^{-j}.$$

Thus, our result will follow if we show that $\sum_i \phi_i (\chi(t^{-1})) = \sum_i \phi_i (\chi(t^{-j}))$. Since $K(\zeta)$ is a splitting field for T , we have that $\chi(t)$ is a root of unity of order dividing $\exp(T)$, so we may assume that $\chi(t) = \xi$, where ξ is some power of ζ .

By (ii), there exists a map $\sigma \in \text{Gal}(K(\zeta) : K)$ such that $\sigma(\zeta) = \zeta^j$, which we extend in the natural way to an automorphism of $K(\zeta)T$ and denote also by σ . Since $\{\phi_i\}_{1 \leq i \leq r} = \{\phi_i \circ \phi\}_{1 \leq i \leq r}$, we have that:

$$\sum_i \phi_i (\chi(t^{-1})) = \sum_i \phi_i (\bar{\xi}) = \sum_i \phi_i \circ \phi (\bar{\xi}) = \sum_i \phi_i (\xi^{-j}) = \sum_i \phi_i (\chi(t^{-j}))$$

as we intended to prove.

The case where T is Hamiltonian now follows as in [1].

3. Final comments

We recall that the supercenter of a group G over a field K is defined as the set $S = S_K(G)$ of all elements in G having a finite number of conjugates in $U(KG)$, the group of units of KG . This subgroup was studied in [2] and its description, in the case where $\text{char}(K) = 0$, was obtained using the theorem on central idempotents given above. Though the statement of [2, Theorem C] is correct, it can now be stated in a more precise form:

Theorem 2. *Let K be a field of characteristic 0 and let G be a non-torsion group. Then one of the following statements holds:*

- (i) $S = Z(G)$, the centre of G .
- (ii) $T(S)$ is an abelian group such that for all $t \in T(S)$ and all $x \in G$ we have that $xtx^{-1} = t^j$ for some positive integer j and, if ζ is a

primitive root of unity of order $o(t)$, then there exists an element $\sigma \in \text{Gal}(K(\zeta) : K)$ such that $\sigma(\zeta) = \zeta^j$. Furthermore, if $T(S)$ is infinite, then $T(S) = \mathbb{Z}(q^\infty) \times B$ where q is a prime rational integer, B is finite central in G , $\mathbb{Z}(q^\infty)$ is central in S , $(G, S) \subset \mathbb{Z}(q^\infty)$ and there exists a positive integer k such that K does not contain roots of unity of order q^k .

Also, [3, Theorem 3.2] can now be stated as follows.

Theorem 3. *Let G be a nilpotent or FC group and let K be a field of characteristic 0. Then $TU(KG)$ is a subgroup if and only if the following conditions hold:*

- (i) T is abelian.
- (ii) For each $t \in T$ and each $x \in G$, there exists a positive integer j such that $xtx^{-1} = t^j$ and, if ζ is a primitive root of unity of order $o(t)$, then there exists a map $\sigma \in \text{Gal}(K(\zeta) : K)$ such that $\sigma(\zeta) = \zeta^k$.

Theorem 1 was used in [4] to obtain a similar result in a nonassociative context, which should now be stated as follows:

Theorem 4. *Let L be an RA loop with torsion subloop T and let K be a field of characteristic 0. Then $TU(KL)$ is a subloop if and only if the following conditions hold:*

- (i) T is an abelian subgroup.
- (ii) For each $t \in T$ and each $x \in L$, there exists a positive integer i such that $xtx^{-1} = t^i$.
- (iii) For each noncentral element $t \in T$ and each $x \in L$, there exists a positive integer j such that $xtx^{-1} = t^j$ and, if ζ is a primitive root of unity of order $o(t)$, then there exists a map $\sigma \in \text{Gal}(K(\zeta) : K)$ such that $\sigma(\zeta) = \zeta^j$.

The proofs are essentially the same as the original ones, requiring only minor changes.

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