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On a generalized Lagrange's identity characterizing inner product spaces

By CLAUDI ALSINA (Barcelona)

Dedicated to Professor Zoltán Daróczy

Abstract. In a real normed space $(E, \|\cdot\|)$ of dimension 3 we show that the existence of a bi-additive function F from $E \times E$ into E, satisfying the generalized Lagrange's identity

$$\rho'_+\left(F(x,y), F(z,v)\right) = \rho'_+(x,z)\rho'_+(y,v) - \rho'_+(x,v)\rho'_+(y,z),$$

where $\rho'_+(a,b)$ is $\|a\|$ multiplied by the right derivative of the norm, implies that the norm must be induced by an inner product.

In three dimensional inner product space $(\mathbb{R}^3, \langle\cdot\,,\cdot\rangle)$ one has the cross product $\times$ satisfying, among others, the bi-additive conditions

$$x \times (y+z) = x \times y + x \times z \quad \text{and} \quad (x+y) \times z = x \times z + y \times z$$

and the well-known Lagrange's identity

$$\langle x \times y, z \times v\rangle = \langle x, z\rangle \cdot \langle y, v\rangle - \langle x, v\rangle \langle y, z\rangle.$$

Let us assume that we have a real normed linear space $(E, \|\cdot\|)$ and the right derivative of the norm $\rho'_+(x,y) = \lim_{t\to 0^+}(\|x+ty\|^2 - \|x\|^2)/2t$ (functional that coincides with the inner product when the norm is induced by it). The norm derivatives play a crucial role in characterizations of inner

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product spaces (see, for example [1], [2], [3], [4]). Our main concern in this paper is to study when it is possible to have in a real normed space $(E, \|\cdot\|)$ an operation F satisfying the bi-additivity conditions and the generalized Lagrange's identity

$$\rho'_+(F(x,y),F(z,v)) = \rho'_+(x,z)\rho'_+(y,v) - \rho'_+(x,v)\rho'_+(y,z).$$

The surprising result is that for dimension 3 the existence of such operations F forces the space to be an inner product space.

Precisely, let $(E, \|\cdot\|)$ be a real normed linear space and consider the norm derivatives defined by

$$\rho'_\pm(x,y) = \lim_{t\to 0^\pm} \frac{\|x+ty\|^2 - \|x\|^2}{2t},$$

for every pair $x, y \in E$. The following properties of $\rho'_\pm$ are well-known (see [1]) and will be used in this paper:

(i) $\rho'_\pm(x,x) = \|x\|^2$ for all $x \in E$;

(ii) $\rho'_\pm(ax,by) = ab\rho'_\pm(x,y)$ if $a \cdot b \geq 0$ and $x, y \in E$;

(iii) $|\rho'_\pm(x,y)| \leq \|x\| \cdot \|y\|$ for all $x, y \in E$;

(iv) $\rho'_\pm(x,ax+y) = a\|x\|^2 + \rho'_\pm(x,y)$ if a is any real and $x, y \in E$;

(v) $\rho'_+(\cdot\,,\cdot)$ is continuous and subadditive in the second variable and $\rho'_-(\cdot\,,\cdot)$ is continuous and superadditive in the second variable and, moreover $\rho'_-(x,y) \leq \rho'_+(x,y)$, for all x, y in E;

(vi) If the norm $\|\cdot\|$ is induced by an inner product $\langle\cdot\,,\cdot\rangle$, then $\rho'_+(x,y) = \rho'_-(x,y) = \langle x,y\rangle$, for all x, y in E.

Let us mention that $\rho'_+(x,y) = \rho'_+(y,x)$ for all x, y in normed space $(E, \|\cdot\|)$ if and only if the norm derives from an inner product, i.e., very weak conditions on $\rho'_\pm$ may characterize inner products.

Our aim in this paper is to determine in a real normed linear space $(E, \|\cdot\|)$ functions F from $E \times E$ into E satisfying the following conditions for all x, y, z, v in E:

$$F(x,y+z) = F(x,y) + F(x,z), \tag{1}$$

$$F(x+y,z) = F(x,z) + F(y,z), \tag{2}$$

and

$$\rho'_+(F(x,y),F(z,v)) = \rho'_+(x,z)\rho'_+(y,v) - \rho'_+(x,v)\rho'_+(y,z). \tag{3}$$

Note that, in particular, (3) implies taking $z = x$ and $v = y$ that

$$\|F(x,y)\|^2 = \|x\|^2\|y\|^2 - \rho'_+(x,y)\rho'_+(y,x) \tag{4}$$

Lemma 1. *If F satisfies* (1), (2) *and* (4) *then*

(6) (i) $F(x,x) = 0$, *for all x in E;*

(7) (ii) $F(y,x) = -F(x,y)$, *for all x,y in E;*

(8) (iii) $F(x, ay+bz) = aF(x,y) + bF(x,z)$, *for all real a,b and for all x,y,z in E.*

Proof. The substitution $y = x$ into (4) yields (i). Next by (i) $F(x+y, x+y) = 0$ and by (1) and (2) one gets (ii). Finally by (4) and the properties of ρ'_+, $F(x,\cdot)$ is continuous at $y = 0$ and by (1) condition (iii) follows.

Lemma 2. *If F satisfies* (1), (2) *and* (3) *then:*

$$\rho'_+(x,y) = \rho'_-(x,y) \quad \text{for all } x,y \text{ in } E. \tag{9}$$

Proof. By (3) and Lemma 1 we have

$$\begin{aligned} 0 &= \rho'_+(F(x,-y),F(y,-y)) = \rho'_+(x,y)\rho'_+(-y,-y) - \rho'_+(x,-y)\rho'_+(-y,y) \\ &= \rho'_+(x,y)\|y\|^2 - (-\rho'_-(x,y))(-\rho'_-(y,y)) = (\rho'_+(x,y) - \rho'_-(x,y))\|y\|^2, \end{aligned}$$

whence for $y \neq 0$, $\rho'_+(x,y) = \rho'_-(x,y)$ and since this last equality is obvious for $y = 0$ we can conclude (9).

Now we prove our main result

Theorem 1. *If $(E, \|\cdot\|)$ is a real normed linear space of dimension 3 and there exists a function F from $E \times E$ into E satisfying* (1), (2) *and* (3) *then necessarily the norm $\|\cdot\|$ is induced by an inner product.*

Proof. Assume that F from $E \times E$ into E satisfies (1), (2) and (3). By the previous lemmas for all x,z in E and a,b in $\mathbb{R}$ we have

$$\|F(z, x+bz)\|^2 = \|F(z, x+az+bz)\|^2,$$

i.e.,

$$\begin{aligned}\|z\|^2\|x+bz\|^2 - \rho'_+(z,x+bz)\rho'_+(x+bz,z)\\ = \|z\|^2\|x+(a+b)z\|^2 - \rho'_+(z,x+(a+b)z)\rho'_+(x+(a+b)z,z)\end{aligned} \tag{10}$$

We can rewrite (10) in the form

$$\begin{aligned}\|z\|^2\left[\|x+(a+b)z\|^2 - \|x+bz\|^2\right] = \rho'_+(z,x+(a+b)z)\\ \times\rho'_+(x+(a+b)z,z) - \rho'_+(z,x+bz)\rho'_+(x+bz,z).\end{aligned} \tag{11}$$

Let us fix x and z two independent vectors in E and let us introduce the function f from $\mathbb{R}$ into $\mathbb{R}$ defined by

$$f(t) = \rho'_+(x+tz,z) = \rho'_-(x+tz,z). \tag{12}$$

Thus by means of (11) and (12) we can write

$$\begin{aligned}&\|z\|^2\left[\|x+(a+b)z\|^2 - \|x+bz\|^2\right]\\ &= \left[(a+b)\|z\|^2 + \rho'_+(z,x)\right] f(a+b) - \left[b\|z\|^2 + \rho'_+(z,x)\right] f(b)\\ &= \left[b\|z\|^2 + \rho'_+(z,x)\right]\left[f(a+b) - f(b)\right] + a\|z\|^2 f(a+b).\end{aligned} \tag{13}$$

Since the norm is continuous and $|f(a+b)| \le \|x+(a+b)z\|\,\|z\|$, taking limits in (13) when $a \to 0\pm$ we obtain

$$\left[b\|z\|^2 + \rho'_+(z,x)\right]\lim_{a\to 0\pm}(f(a+b) - f(b)) = 0,$$

i.e., for any real b, $b \ne b_0 := -\rho'_+(z,x)/\|z\|^2$, we obtain

$$\lim_{a\to 0\pm} f(b+a) = f(b). \tag{14}$$

Note that by (13) at point b_0 we have

$$\begin{aligned}\lim_{a\to 0\pm} f(b_0+a) &= \lim_{a\to 0^\pm}\frac{\|x+b_0z+az\|^2 - \|x+b_0z\|^2}{a}\\ &= 2\rho'_\pm(x+b_0z,z) = 2f(b_0).\end{aligned} \tag{15}$$

We claim that $f(b_0) = 0$. To see this consider the following chain of equalities for any real λ and for our fixed x, z:

$$\begin{aligned}
\lambda^2\|F(x,z)\|^2 &= \|F(x,\lambda z)\|^2 = \|F(x+b_0z,\lambda z)\|^2 \\
&= \|F(x+b_0z, x+b_0z+\lambda z)\|^2 = \|x+b_0z\|^2\cdot\|x+b_0z+\lambda z\|^2 \\
&\quad - \rho'_+(x+b_0z, x+b_0z+\lambda z)\rho'_+(x+b_0z+\lambda z, x+b_0z+\lambda z-\lambda z) \\
&= \|x+b_0z\|^2\|x+b_0z+\lambda z\|^2 \\
&\quad - \left[\|x+b_0z\|^2 + \rho'_+(x+b_0z,\lambda z)\right]\left[\|x+b_0z+\lambda z\|^2\right. \\
&\quad \left.+ \rho'_+(x+b_0z+\lambda z, -\lambda z)\right] \\
&= -\|x+b_0z\|^2\rho'_+(x+(b_0+\lambda)z, -\lambda z) \\
&\quad - \|x+b_0z+\lambda z\|^2\rho'_+(x+b_0z,\lambda z) \\
&\quad - \rho'_+(x+b_0z,\lambda z)\rho'_+(x+(b_0+\lambda)z,-\lambda z)).
\end{aligned} \tag{16}$$

Then taking into account that we have already proved that in our case $\rho'_+ = \rho'_-$, division by $\lambda < 0$ in (16) yields

$$\begin{aligned}
\lambda\|F(x,z)\|^2 &= +\|x+b_0z\|^2\rho'_+(x+(b_0+\lambda)z,z) \\
+\|x+b_0z+\lambda z\|^2\rho'_+(x+b_0z,z) &- \lambda\rho'_+(x+b_0z,z)\rho'_+(x+(b_0+\lambda)z,z)
\end{aligned} \tag{17}$$

and taking limits when $\lambda \to 0-$ we obtain using (15)

$$0 = \|x+b_0z\|^2 2f(b_0) + \|x+b_0z\|^2 f(b_0),$$

and since x and z are independent, $f(b_0) = 0$. Therefore by (17), for any $\lambda < 0$ it is

$$\lambda\|F(x,z)\|^2 = \|x+b_0z\|^2 f(b_0+\lambda)$$

i.e., for any $t < b_0$:

$$f(t) = f(b_0+(t-b_0)) = \frac{\|F(x,z)\|^2}{\|x+b_0z\|^2}(t-b_0),$$

so f is an affine function on $(-\infty, b_0]$. Since $f(b_0) = 0$ by (16) we also have for $\lambda > 0$

$$\lambda^2\|F(x,z)\|^2 = \lambda\|x+b_0z\|^2 f(b_0+\lambda),$$

i.e., for any $t > b_0$:

$$f(t) = f(b_0 + (t - b_0)) = \frac{\|F(x,z)\|^2}{\|x + b_0 z\|^2}(t - b_0),$$

so f is an affine function on $\mathbb{R}$ vanishing at b_0. Thus for all real t

$$\rho'_+(x + tz, z) = \frac{\|F(x,z)\|^2}{\|x - \frac{\rho'_+(z,x)}{\|z\|^2} z\|^2} \left(t + \frac{\rho'_+(z,x)}{\|z\|^2} \right),$$

and for $t = 0$ we obtain:

$$\rho'_+(x,z) = \frac{\|x\|^2 \|z\|^2 - \rho'_+(x,z)\rho'_+(z,x)}{\|x - \frac{\rho'_+(z,x)}{\|z\|^2} z\|^2} \cdot \frac{\rho'_+(z,x)}{\|z\|^2},$$

i.e., $\rho'_+(x,z) = 0$ if and only if $\rho'_+(z,x) = 0$ and the symmetry of the orthogonality relation $\rho'_+(x,z) = 0$ yields that necessarily (in dimension 3) the norm derives from an inner product (see [1]). This completes the proof.

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CLAUDI ALSINA
UNIVERSITAT POLITÈCNICA DE CATALUNYA
ETSAB. DIAGONAL 649
E–08028 BARCELONA
SPAIN

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