

On generalized q -multiplicative functions

By J. FEHÉR

*Dedicated to the 60th birthday
of Professors Zoltán Daróczy and Imre Káta*

Abstract. The $\mathcal{R}$ -multiplicative functions are defined as a generalization of q -multiplicative functions. Those $\mathcal{R}$ -multiplicative functions are investigated for which a linear recurrence holds.

1. Introduction

The letters $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{R}$, $\mathbb{C}$ denote the sets of the natural numbers, nonnegative integers, real numbers, complex numbers, respectively. For $A \subseteq \mathbb{N}_0$ let $|A|$ be the number of the elements of A .

Definition 1.1. Let $\mathcal{R}_0, \mathcal{R}_1, \dots$ be a sequence of subsets of $\mathbb{N}_0$. We say that it is an $\mathcal{R}$ -system, if the following conditions hold:

- a) $0 \in \mathcal{R}_i$ and $1 < |\mathcal{R}_i| < \infty$ ($i = 0, 1, 2, \dots$);
- b) for $(0 \leq) i < j$, the smallest positive element of $\mathcal{R}_i$ is smaller than the smallest positive element of $\mathcal{R}_j$;
- c) each $n \in \mathbb{N}_0$ can be uniquely written as

$$(1.1) \quad n = \sum_{j=0}^s r_j \quad (r_j \in \mathcal{R}_j, \ s \geq 0).$$

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We say that an $\mathcal{R}$ -system is monotonic, if in addition (d), and that it is bounded if in addition (e) holds, where:

- d) for each $(0 \leq) i < j$, the largest element of $\mathcal{R}_i$ is smaller than the smallest positive element of $\mathcal{R}_j$,
- e) $|\mathcal{R}_j|$ is bounded.

Examples 1.2. Let k_i ($i = 0, 1, \dots$) be a sequence of integers, $k_i \geq 2$, furthermore let $d_0 = 1$, $d_i = d_{i-1}k_{i-1}$ ($i > 0$), $\mathcal{N}_i = \{0, 1, \dots, k_i - 1\}$, $\mathcal{R} = d_i\mathcal{N}_i = \{0, d_i, \dots, (k_i - 1)d_i\}$ ($i = 0, 1, \dots$). Then $\mathcal{R}_i$ ($i = 0, 1, \dots$) is an $\mathcal{R}$ -system. Such $\mathcal{R}$ -systems are called “britannic number systems” by N. G. De BRUIJN [1].

Especially, if $k_i = q \geq 2$ ($i = 0, 1, \dots$) then we obtain the q -ary number system. It is easy to see that the monotonic $\mathcal{R}$ -systems are exactly the “britannic number systems”.

Definition 1.3. The function $f : \mathbb{N}_0 \rightarrow \mathbb{C}$ is called $\mathcal{R}$ -multiplicative (with respect to a given $\mathcal{R}$ -system), if

$$(1.2) \quad f(0) = 1 \text{ and } f(n) = \prod_{j=0}^s f(r_j).$$

It is clear that $f(n) = c^n$ ($0 \neq c \in \mathbb{C}$) is $\mathcal{R}$ -multiplicative for every $\mathcal{R}$ -system.

2. $\mathcal{R}$ -multiplicative functions with regular behaviour

Let an $\mathcal{R}$ -system be given, $f : \mathbb{N}_0 \rightarrow \mathbb{C}$ be an $\mathcal{R}$ -multiplicative function, $P(z) = a_k z^k + \dots + a_1 z + a_0 \in \mathbb{C}[z]$, and

$$P(E)f(N) := a_k f(n+k) + \dots + a_1 f(n+1) + a_0 f(n).$$

Let us consider the following conditions:

$$(2.1) \quad \liminf_{x \rightarrow \infty} \frac{1}{x} \sum_{n \leq x} |P(E)f(n)| = 0,$$

$$(2.2) \quad \sum_{n \leq x} |P(E)f(n)| = o(x) \quad (x \rightarrow \infty),$$

$$(2.3) \quad P(E)f(n) = 0 \quad (\forall n \in \mathbb{N}_0),$$

$$(2.4) \quad \liminf_{x \rightarrow \infty} \frac{1}{x} \sum_{n \leq x} |f(n)| = 0,$$

$$(2.5) \quad \sum_{n \leq x} |f(n)| = o(x) \quad (x \rightarrow \infty).$$

Theorem 2.1. *The following assertions are valid:*

- a) *If (2.1) holds then either (2.3) or (2.4) hold*
- b) *(2.2) is satisfied if and only if either (2.3) or (2.5) holds.*
- c) *There exists such an f for which (2.4) holds, and (2.5) does not hold.*
- d) *Assuming that the $\mathcal{R}$ -system is monotonic and bounded, the fulfilment of (2.4) and that of $|f(n)| \leq 1$ ($n \in \mathbb{N}_0$) imply (2.5).*

PROOF. Assume that the $\mathcal{R}$ -system is given: $\mathcal{R} = \{\mathcal{R}_0, \mathcal{R}_1, \dots\}$. Let the sets $\mathcal{A}_s, \mathcal{T}_s$ be defined as follows: $\mathcal{A}_0 = \emptyset$, $\mathcal{A} = \mathcal{R}_0 \oplus \mathcal{R}_1 \oplus \dots \oplus \mathcal{R}_{s-1}$ ($s \geq 1$), $\mathcal{T}_s = \bigcup_{\ell=0}^{\infty} (\mathcal{R}_s \oplus \dots \oplus \mathcal{R}_{s+\ell})$, i.e. $\mathcal{T}_s$ consists of those integers n which can be written as $n = r_s + \dots + r_{s+\ell}$ for some integer ℓ , and $r_j \in \mathcal{R}_j$ ($j = s, \dots, s + \ell$).

It is clear that $f(n+m) = f(n)f(m)$ is satisfied if $n \in \mathcal{A}_s$, $m \in \mathcal{T}_s$ and f is $\mathcal{R}$ -multiplicative.

- a) Let f be $\mathcal{R}$ -multiplicative, assume that (2.1) holds, and that for some $\alpha \in \mathbb{N}_0$ $P(E)f(\alpha) \neq 0$. Let i_0 be so large that $\alpha + j \in \mathcal{A}_{i_0}$ for $j = 0, 1, \dots, k$, where $k = \deg P$. Let $\mathcal{A}_{i_0} = \{\beta_1, \beta_2, \dots, \beta_A\}$. Then for every large x ,

$$(2.6) \quad \frac{1}{x} \sum_{n \leq x} |P(E)f(n)| \geq \frac{x-\alpha}{x} |P(E)f(\alpha)| \frac{1}{x-\alpha} \sum_{\substack{n \leq x-\alpha \\ n \in \mathcal{T}_{i_0}}} |f(n)| \geq 0.$$

The relation (2.1) and (2.6) imply that for an appropriate sequence $y_t \rightarrow \infty$ ($t \rightarrow \infty$) we get

$$(2.7) \quad \frac{1}{y_t} \sum_{\substack{n \leq y_t \\ n \in \mathcal{T}_{i_0}}} |f(n)| \rightarrow 0 \quad (t \rightarrow \infty).$$

Furthermore, for large y ,

$$\sum_{n \leq y} |f(n)| = \sum_{j=1}^A |f(\beta_j)| \left(\sum_{\substack{n+\beta_j \leq y \\ n \in \tau_{i_0}}} |f(n)| \right) \leq A \cdot H \sum_{\substack{n \leq y \\ n \in \tau_{i_0}}} |f(n)|,$$

where $H = \max_{1 \leq j \leq A} |f(\beta_j)|$. Hence, and from (2.7),

$$(2.8) \quad \frac{1}{y_t} \sum_{n \leq y_t} |f(n)| \rightarrow 0 \quad (t \rightarrow \infty).$$

- b) Assume that for some $\mathcal{R}$ -multiplicative function f (2.2) holds, and that there exists an $\alpha \in \mathbb{N}_0$ such that $P(E)f(\alpha) \neq 0$. We can choose x_t to run over the whole set $\mathbb{N}_0$ and reason as above.
- c) We shall give an $\mathcal{R}$ -system and an $\mathcal{R}$ -multiplicative function for which (2.4) holds and (2.5) does not hold. Let f be a q -multiplicative function taking on positive values. Let

$$H(s) := \frac{1}{q^s} \sum_{n=0}^{q^s-1} f(n) = \prod_{i=0}^{s-1} \left(\frac{1}{q} \sum_{j=0}^{q-1} f(jq^i) \right).$$

Let $s_1 < s_2 < \dots$ be a rare sequence of integers, let $f(jq^t) = \frac{1}{2q^2}$ if $j = 1, 2, \dots, q-1$; $t \in \mathbb{N}_0$ except for $j = 1$, when $t \in \{s_1, s_2, \dots\}$. Let $f(1 \cdot q^{s_\ell}) = q^{s_\ell} + 1$. Then obviously,

$$\frac{1}{q^{s_\ell} + 1} \sum_{n=0}^{q^{s_\ell}} f(n) > 1,$$

on the other hand

$$\frac{1}{q} \sum_{j=0}^{q-1} f(jq^\ell) = \begin{cases} \frac{1}{q} + \frac{q-1}{2q} & \text{if } \ell \notin \{s_1, s_2, \dots\} \\ q^\ell + \frac{2}{q} + \frac{q-1}{2q^2} < q^m + 1 & \text{if } \ell \in \{s_1, s_2, \dots\}. \end{cases}$$

Thus

$$H(s) \leq \left(\frac{3}{2q} \right)^s \prod_{s_\nu \leq s} (q^{s_\nu} + 1) \left(\frac{2q}{3} \right).$$

If we choose $s_\nu = 2^{2^\nu}$, and $t_\nu = s_{\nu+1} - 1$, then

$$H(t_\nu) \leq \left(\frac{3}{2q}\right)^{t_\nu - 1 - \nu} q^{2s_\nu + 1} \rightarrow 0 \quad (\nu \rightarrow \infty).$$

- d) To prove it, we may assume that f takes on nonnegative values, $0 \leq f(n) \leq 1$. Let n_s be a strictly monotonic sequence of positive integers, and $n_s - 1 = Ad_{j_s} + n'_s$, where $1 \leq A < k_{j_s}$, $0 \leq n'_s < d_{j_s}$. Then

$$(2.9) \quad \frac{1}{n_s} \sum_{m=0}^{n_s-1} f(m) \geq \frac{d_{j_s}}{n_s} \cdot \frac{1}{d_{j_s}} \sum_{m=0}^{d_{j_s}-1} f(m) \geq 0.$$

Let n_s be such a sequence for which the right hand side of (2.9) tends to zero. Since the sequence k_i in the definition of the $\mathcal{R}$ -system is assumed to be bounded, $k_i \leq M$, therefore

$$\frac{d_{j_s}}{n_s} \geq \frac{d_{j_s}}{d_{j_s+1}} = \frac{1}{k_{j_s}} \geq \frac{1}{M},$$

consequently from (2.9)

$$(2.10) \quad \frac{1}{d_{j_s}} \sum_{m=0}^{d_{j_s}-1} f(m) \rightarrow 0 \quad (s \rightarrow \infty).$$

It is obvious that for every $h \geq 1$,

$$(2.11) \quad \frac{1}{d_h} \sum_{m=0}^{d_h-1} f(m) = \prod_{i=0}^{h-1} \left(\frac{1}{k_i} \sum_{t \in \mathcal{R}_i} f(t) \right) =: A(h).$$

Since $0 < \frac{1}{k_i} \sum_{t \in \mathcal{R}_i} f(t) \leq 1$, the sequence $A(h)$ is decreasing monotonically, and by (2.10) we obtain that $A(h) \rightarrow 0$ ($h \rightarrow \infty$).

Let $n > 1$, $n - 1 = Ad_{i(n)} + n'$ ($0 \leq A < k_{i(n)}$, $0 \leq n' < d_{i(n)}$). Then

$$(2.12) \quad \begin{aligned} 0 &\leq \frac{1}{n} \sum_{m=0}^{n-1} f(m) \leq \frac{1}{n} \sum_{j=0}^A f(jd_{i(n)}) \sum_{m=0}^{d_{i(n)}-1} f(m) \leq \\ &\leq (A+1) \frac{d_{i(n)}}{n} A(i(n)) \leq 2A(i(n)). \end{aligned}$$

Hence the assertion follows.

The proof of Theorem 2.1 is complete.

Theorem 2.2. *Assume that the $\mathcal{R}$ -system is monotonic, and that $F : \mathbb{N}_0 \rightarrow \mathbb{R}$ is such an $\mathcal{R}$ -multiplicative function for which $0 \leq F(n) \leq 1$ ($\forall n \in \mathbb{N}_0$). Then*

$$(2.13) \quad \sum_{n \leq x} F(n) = o(x) \quad (x \rightarrow \infty)$$

holds if and only if

$$(2.14) \quad \sum_{j=0}^{\infty} \frac{1}{k_j} \left(k_j - \sum_{r \in \mathcal{R}_j} F(r) \right) = \infty.$$

PROOF. From (2.11), (2.12) it follows that (2.13) holds if and only if

$$(2.15) \quad \prod_{j=0}^{\infty} \left(\frac{1}{k_j} \sum_{r \in \mathcal{R}_j} F(r) \right) \rightarrow 0.$$

Since

$$\frac{1}{k_j} \sum_{r \in \mathcal{R}_j} F(r) = 1 - \frac{1}{k_j} \left(k_j - \sum_{r \in \mathcal{R}_j} F(r) \right),$$

and

$$k_j - \sum_{r \in \mathcal{R}_j} F(r) \geq 0,$$

(2.15) holds if and only if (2.14) is satisfied.

3. The $\mathcal{R}$ -multiplicative solutions of the recursion

$$P(E)f(n) = 0$$

Theorem 3.1. *Let $\mathcal{R}_0, \mathcal{R}_1, \dots$ be such an $\mathcal{R}$ -system for which $\mathcal{R}_0 = \{0, 1, \dots, d-1\}$, $d \geq 2$. Assume that $P(z) \in \mathbb{C}[z]$ is of degree k , $1 \leq k \leq d$,*

and that $P(0) \neq 0$. Then the recursion

$$(3.1) \quad P(E)f(n) = 0 \quad (n \in \mathbb{N}_0)$$

holds for an $\mathcal{R}$ -multiplicative function if and only if

$$(3.2) \quad f(n) = \sum_{j=1}^s \alpha_j \rho_j^n,$$

where $\sum_{j=1}^s \alpha_j = 1$, and $\rho_j (j = 1, 2, \dots, s)$ are distinct roots for P for which

$$(3.3) \quad \rho_1^d = \rho_2^d = \dots = \rho_s^d \quad (= c).$$

PROOF. First we prove that the function f defined by (3.2) is $\mathcal{R}$ -multiplicative. $f(0) = \alpha_1 + \dots + \alpha_s = 1$. Let $N, r \in \mathbb{N}_0$. From (3.2), (3.3) we obtain that $f(Nd+r) = C^N(\alpha_1 \rho_1^r + \dots + \alpha_s \rho_s^r) = C^N f(r) = f(Nd)f(r)$, from which the $\mathcal{R}$ -multiplicativity of f is clear. The fulfilment of (3.1) is obvious.

Assume now that f is an $\mathcal{R}$ -multiplicative function and that (3.1) holds. Then

$$\begin{aligned} 0 &= P(E)f(Nd + d - k) \\ &= a_k(f((N+1)d) - f(Nd)f(d)) + f(Nd)P(E)f(d - k). \end{aligned}$$

Hence, by (3.2) and from $a_k \neq 0$ we have that

$$(3.4) \quad f((N+1)d) = f(Nd)f(d).$$

From $f(d) = 0$ it would follow that $f(0) = 0$, which cannot be. Thus

$$(3.5) \quad f(Nd) = C^N \quad (\forall N \in \mathbb{N}_0), \quad 0 \neq C \in \mathbb{C}.$$

The general solution of (3.1) can be written as

$$(3.6) \quad f(n) = p_1(n)\rho_1^n + \dots + p_h(n)\rho_h^n,$$

where ρ_j are the roots of $P(z)$ and p_j are polynomials.

(3.5), (3.6) and the $\mathcal{R}$ -multiplicativity of f imply that

$$(3.7) \quad \sum_{i=1}^h p_i(Nd+r) \sigma_i^N \rho_i^r - f(r) c^N = 0$$

holds for every $N \in \mathbb{N}_0$ and $r = 0, 1, \dots, d-1$. Here $\sigma_j := \rho_j^d$ ($j = 1, \dots, h$).

Since $f(0) = 1$, from (3.7) by $r = 0$ we obtain that $\sigma_i = C$ holds for at least one i .

Assume that $\sigma_1 = \dots = \sigma_{i_1}$, $\sigma_{i_1+1} = \dots = \sigma_{i_2}$, $\dots$, $\sigma_{i_{t-1}+1} = \dots = \sigma_{i_t}$, $\sigma_{i_t+1} = \dots = \sigma_h = C$, and that $\sigma_{i_\nu} \neq \sigma_{i_\mu}$ if $\nu \neq \mu$. Assume that $i_1 < h$.

Let

$$Q_{i_j}^{(r)}(N) := p_{i_{j-1}+1}(Nd+r) \rho_{i_{j-1}+1}^r + \dots + p_{i_j}(Nd+r) \rho_{i_j}^r \\ (j = 1, \dots, t; i_0 := 0),$$

$$Q^{(r)}(N) := p_{i_t+1}(Nd+r) \rho_{i_t+1}^r + \dots + p_h(Nd+r) \rho_h^r - f(r).$$

By using these notations (3.7) can be rewritten as

$$(3.8) \quad Q_{i_1}^{(r)}(N) \sigma_{i_1}^N + \dots + Q_{i_t}^{(r)}(N) \sigma_{i_t}^N + Q^{(r)}(N) C^N = 0 \quad (N \in \mathbb{N}_0).$$

Consequently the polynomials $Q_{i_\nu}^{(r)}(x)$, $Q^{(r)}(x)$ are zero identically, i.e.

$$(3.9) \quad p_{i_{j-1}+1}(z) \rho_{i_{j-1}+1}^r + \dots + p_{i_j}(z) \rho_{i_j}^r = 0 \\ (r = 0, 1, \dots, i_j - i_{j-1} - 1 < d)$$

for $z \in \mathbb{C}$, since $\rho_{i_{j-1}+1}, \dots, \rho_{i_j}$ are distinct complex numbers, the determinant of the system, being a Vandermonde determinant, is nonzero, hence $p_\nu(z) = 0$ identically, and consequently

$$(3.10) \quad f(n) = q_1(n) \rho_1^n + \dots + q_s(n) \rho_s^n,$$

where $1 \leq s \leq h$, $\rho_1^d = \dots = \rho_s^d = C$, $q_1, \dots, q_s \in \mathbb{C}[x]$.

From (3.5) and from the $\mathcal{R}$ -multiplicativity of f ,

$$f(r) = q_1(Nd+r) \rho_1^r + \dots + q_s(Nd+r) \rho_s^r \quad (N \in \mathbb{N}_0, 0 \leq r < d).$$

Hence we obtain easily that the coefficients $q_j(Nd+r)$ do not depend on N , consequently $q_j(Nd+r) = \alpha_j$ ($j = 1, \dots, s$).

Finally, from $f(0) = 1$ we obtain that $\alpha_1 + \dots + \alpha_s = 1$. The proof of Theorem 3.1 is complete.

4. Remark to the Theorem 3.1

To prove (3.1) the assumption of the condition $\deg P \leq d$ is important. If $\deg P > d$, then in general there exists such an $\mathcal{R}$ -multiplicative function for which (3.1), (3.2) are satisfied, but (3.3) does not hold.

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J. FEHÉR
JANUS PANNONIUS UNIVERSITY
IFJUSÁG U. 6.
7624 PÉCS
HUNGARY

E-mail: feher@math.jpte.hu

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