On positive definite quadratic forms.

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The well known necessary and sufficient condition for a quadratic form to be positive definite has been proved recently by three authors, Á. CSÁSZÁR [1], T. SZELE [2] and E. EGERVÁRY [3]. In this note another proof is given. This involves only the elementary theory of matrices, and is the shortest of all four proofs.

Consider the quadratic form, $\mathbf{x}'A\mathbf{x}$, where $\mathbf{x}$ is the column-vector $\{x_1, \ldots, x_n\}$, $\mathbf{x}'$ is the transpose of $\mathbf{x}$, that is the row vector $(x_1, \ldots, x_n)$, and A is a symmetric matrix of order n over the real field. Denote by A_r the matrix consisting of the elements of A in the first r rows and first r columns, $(A_0 = 1)$, and by $|A|$ the determinant of A.

Lemma 1: *If $\mathbf{x}'A\mathbf{x}$ is positive definite, the matrix A is non-singular.*

This follows from the fact that if A is singular there exists a non-zero vector, $\mathbf{c}$, such that $A\mathbf{c} = 0$. Then $\mathbf{c}'A\mathbf{c} = 0$, and $\mathbf{x}'A\mathbf{x}$ is therefore not positive definite.

Lemma 2: *If $|A_i| \neq 0 \quad (i = 1, 2, \ldots, n)$ there exists a unit lower triangular matrix, L, of the form*

$$L = \begin{pmatrix} 1 & & & \\ l_{21} & 1 & & \\ \vdots & & \ddots & \\ l_{n1} & \ldots & l_{n,n-1} & 1 \end{pmatrix}$$

such that

$$A = L'DL,$$

where

$$D = \begin{pmatrix} \delta_1 & & & \\ & \delta_2 & & \\ & & \ddots & \\ & & & \delta_n \end{pmatrix}$$

and $\delta_i = |A_i| / |A_{i-1}|$.

This lemma may be verified by direct computation. A proof has been outlined by TURING [4], p. 289.

We now have the

Theorem: *The necessary and sufficient condition that* $\mathbf{x}'A\mathbf{x}$ *be positive definite is that*

$$|A_i| > 0 \qquad (i = 1, 2, \ldots, n).$$

Proof: *Sufficiency.* Since $|A_i| > 0$ $(i = 1, 2, \ldots, n)$ we have, by lemma 2,

$$A = L'DL.$$

Now

$$\mathbf{x}'A\mathbf{x} = (\mathbf{x}'L')\,D(L\mathbf{x}) = \mathbf{y}'D\mathbf{y} = \delta_1 y_1^2 + \delta_2 y_2^2 + \cdots + \delta_n y_n^2,$$

where

$$\mathbf{y} = L\mathbf{x} = \{y_1, \ldots, y_n\}.$$

Since $\delta_1, \ldots, \delta_n$ are all positive, $\mathbf{y}'D\mathbf{y}$ and therefore $\mathbf{x}'A\mathbf{x}$ is positive definite,

Necessity. Let $\mathbf{x}_r = \{x_1, \ldots, x_r\}$. Since $\mathbf{x}'A\mathbf{x}$ is positive definite it follows, by considering the vector $\{x_1, \ldots, x_r, 0, \ldots, 0\}$, that $\mathbf{x}_r'A_r\mathbf{x}_r$ is positive definite. Hence, by lemma 1, $|A_r| \neq 0$. This is true for $r = 1, \ldots, n$. Thus, by lemma 2. $A = L'DL$, and, as before,

$$\mathbf{x}'A\mathbf{x} = \delta_1 y_1^2 + \cdots + \delta_n y_n^2 = \mathbf{y}'D\mathbf{y},$$

where $\mathbf{y} = L\mathbf{x}$. But $\mathbf{x}'A\mathbf{x}$, and hence $\mathbf{y}'D\mathbf{y}$, is positive definite. Hence $\delta_i > 0$ $(i = 1, \ldots, n)$. and thus

$$|A_i| > 0 \qquad (i = 1, \ldots, n).$$

References.

[1] Á. Császár, Sur les formes quadratiques positives, *Publicationes Mathematicae (Debrecen)* **1** (1950), 186—188.

[2] T. Szele, Über die Klassifikation der quadratischen Formen, *Publicationes Mathematicae (Debrecen)* **1** (1950), 189—192.

[3] E. Egerváry, Eine Bemerkung über definite quadratische Formen, *Publicationes Mathematicae (Debrecen)* **1** (1950), 193—195.

[4] A. M. Turing, Rounding- off erros in matrix processes, *The Quarterly Journal of Mechanics and Applied Mathematics,* **1** (1948), 287—308.

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