

On integral representations for powers of the Riemann zeta-function

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*Dedicated to Professor Imre Kátai
on the occasion of his 60th birthday*

Abstract. A new integral representation for $\zeta^r(s)$ is obtained, where $r \geq 3$ is a fixed natural number. The approach is due to A. Guthmann, who obtained the analogue of the classical Riemann-Siegel formula (for $\zeta(s)$) for several Dirichlet series, including $\zeta^2(s)$. The fundamental role is played by the Mellin inverse of $\pi^{-rs/2}\Gamma^r(s/2)\zeta^r(s)$. The properties of this function are studied in detail and in particular its asymptotic expansion is given.

1. Introduction

Integral representations of Dirichlet series are a major tool in Analytic Number Theory. Of special prominence is the classical Riemann-Siegel formula (see C.L. SIEGEL [12])

$$(1.1) \quad \begin{aligned} \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) &= \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\int_{0\swarrow 1}\frac{e^{i\pi x^2}x^{-s}}{e^{i\pi x}-e^{-i\pi x}}dx \\ &+ \pi^{(s-1)/2}\Gamma\left(\frac{1-s}{2}\right)\int_{0\searrow 1}\frac{e^{-i\pi x^2}x^{s-1}}{e^{i\pi x}-e^{-i\pi x}}dx. \end{aligned}$$

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This is valid for s not equal to the poles of $\Gamma(s)$, and $0 \swarrow 1$ (resp. $0 \searrow 1$) denotes a straight line which starts from infinity in the upper complex half-plane, has slope equal to 1 (resp. to -1), and cuts the real axis between 0 and 1. The integrals in (1.1) are of a fairly simple nature, and they can be evaluated asymptotically to provide precise formulas for $\zeta(s)$ (see [6], [12], [13] and (7.9)). Although (1.1) has been known for a long time, its direct generalization to other Dirichlet series, which possess functional equations with gamma-factors similar to the functional equation of $\zeta(s)$, remained an open problem. It is only in the early 1980's that Y. MOTOHASHI [8]–[10] obtained the asymptotic expansion of $\zeta^2(s)$. His method, however, uses some intrinsic properties of the function $d(n)$ (the number of divisors of n), and cannot be readily generalized. Also due to some unfortunate circumstances (see the postscript in [10]) a detailed proof of his results was not appropriately published in due time.

It is only recently that A. GUTHMANN devised a general approach for obtaining integral representations for Dirichlet series, which may be regarded as a generalization of the Riemann-Siegel integral formula (1.1). In his Habilitation Thesis [2] and in [4] he obtained an analogue of (1.1) for zeta-functions of holomorphic cusp forms, and in [3] for $\zeta(s)\zeta(s+1)$. In [5] he further developed his ideas to tackle $\zeta^2(s)$. It is the purpose of this paper to obtain an analogous integral representation for $\zeta^r(s)$, where $r \geq 3$ is an arbitrary, but fixed natural number. This in turn depends on properties of the inverse Mellin transform of $\pi^{-rs/2}\Gamma^r(s/2)\zeta^r(s)$. This function, which we shall denote by $\psi_r(x)$, appears to be of intrinsic interest and it will be extensively studied in the sequel. Generalizations of our integral representations for $\zeta^r(s)$ to other Dirichlet series possessing functional equations with multiple gamma-factors are possible.

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2. The outline of the method

For $r \geq 1$ a fixed integer, $c > 0$ and $\Re x > 0$ let

$$(2.1) \quad f_r(x) = \frac{1}{2\pi i} \int_{(c)} \Gamma^r\left(\frac{s}{2}\right) \left(\frac{x}{2}\right)^{-s} ds,$$

where the integral is absolutely convergent and, as usual,

$$\int_{(c)} F(s) ds = \lim_{T \rightarrow \infty} \int_{c-iT}^{c+iT} F(s) ds.$$

Then it follows by the Mellin inversion formula (see the Appendix of [6]) that

$$(2.2) \quad \Gamma^r \left(\frac{s}{2} \right) 2^s = \int_0^\infty f_r(x) x^{s-1} dx \quad (\Re s > 0).$$

If $d_r(n)$ is the number of ways in which $n \in \mathbb{N}$ may be written as a product of r fixed factors ($d_1(n) = 1$, $d_2(n) = d(n)$), then for $\Re s > 1$

$$\zeta^r(s) = \sum_{n=1}^{\infty} d_r(n) n^{-s}.$$

Consequently by absolute convergence we have, for $\Re x > 0$, $c > 1$,

$$(2.3) \quad \psi_r(x) := \sum_{n=1}^{\infty} d_r(n) f_r(2\pi^{\frac{r}{2}} x n) = \frac{1}{2\pi i} \int_{(c)} \pi^{-\frac{rs}{2}} \zeta^r(s) \Gamma^r \left(\frac{s}{2} \right) x^{-s} ds,$$

hence by the Mellin inversion formula we have

$$(2.4) \quad \pi^{-\frac{rs}{2}} \zeta^r(s) \Gamma^r \left(\frac{s}{2} \right) = \int_0^\infty \psi_r(x) x^{s-1} dx \quad (\Re s > 1).$$

The shape of the left-hand side of (2.4) is such that it remains unchanged if s is replaced by $1 - s$. This follows from the symmetric form of the functional equation for $\zeta(s)$ (see [6] or [13]), namely

$$(2.5) \quad \pi^{-\frac{s}{2}} \Gamma \left(\frac{s}{2} \right) \zeta(s) = \pi^{-\frac{(1-s)}{2}} \Gamma \left(\frac{1-s}{2} \right) \zeta(1-s).$$

In fact it is precisely the symmetry furnished by (2.5) which is crucial in deriving integral representations for $\zeta^r(s)$. The function $\psi_r(x)$ is holomorphic for $\Re x > 0$, and it is of exponential decay as $x \rightarrow \infty$. To see this let $X = \pi^r x^2$, $c > 1$. Then from (2.3) we obtain

$$(2.6) \quad \psi_r(x) = \frac{1}{\pi i} \int_{(c)} \zeta^r(2w) \Gamma^r(w) X^{-w} dw.$$

Since e^{-x} , $\Gamma(s)$ is a pair of Mellin transforms, the Parseval identity for Mellin transforms (see (A.5) of [6]) gives

$$(2.7) \quad \int_0^\infty |\Gamma(\sigma + it)|^2 dt = \pi 2^{-2\sigma} \Gamma(2\sigma) \quad (\sigma > 0).$$

But for $\Re w \geq 2$ we have $|\zeta(w)| \leq \frac{\pi^2}{6}$, and for $x > 0$ (see e.g. N.N. LEBEDEV [7])

$$(2.8) \quad |\Gamma(x + iy)| \leq \Gamma(x) = \sqrt{2\pi} x^{x-\frac{1}{2}} e^{-x} (1 + r(x)), \quad |r(x)| \leq e^{\frac{1}{12x}} - 1.$$

Assume now that $r \geq 2$. Since $e^x \leq 1 + 2x$ for $0 \leq x \leq 1$ we have $|1 + r(x)| \leq 7/6$ for $x \geq 1$, hence from (2.6)–(2.8) it follows that, for $x, c \geq 1$ and $r \geq 2$,

$$\begin{aligned} |\psi_r(x)| &\leq \frac{2}{\pi} \left(\frac{\pi^2}{6} \right)^r X^{-c} \int_0^\infty |\Gamma(c + it)|^2 \Gamma^{r-2}(c) dt \\ &= 2 \left(\frac{\pi^2}{6} \right)^r X^{-c} 2^{-2c} \Gamma(2c) \Gamma^{r-2}(c) \\ &\leq \sqrt{2} \left(\frac{\pi^2}{6} \right)^r \left(\frac{7}{6} \sqrt{2\pi} \right)^{r-1} \exp \left(-c \log X + rc \log c - rc + \frac{1-r}{2} \log c \right) \\ &= \frac{6}{7\sqrt{\pi}} \left(\frac{7\pi^{5/2} 2^{1/2}}{36} \right)^r X^{\frac{1-r}{2r}} \exp \left(-rX^{\frac{1}{r}} \right) \end{aligned}$$

with the choice $c = X^{1/r}$ (≥ 1). Therefore we obtain

$$(2.9) \quad |\psi_r(x)| \leq \frac{6}{7} \left(\frac{7\pi^2 2^{1/2}}{36} \right)^r x^{\frac{1-r}{r}} \exp \left(-r\pi x^{\frac{2}{r}} \right) \quad (x \geq 1, r \geq 2),$$

and from (3.2) it is seen that (2.9) also holds when $r = 1$. For an asymptotic expansion of $\psi_r(x)$ when r is fixed, see (5.11). Actually no absolute value signs are needed in (2.9), since $\psi_r(x) > 0$ (and we have $\psi'_r(x) < 0$) for $x > 0$. This follows from the series representation (2.3) and the properties of $f_r(x)$ (see (3.11) for the proof that $f_r(x) > 0$, $f'_r(x) < 0$ when $x > 0$).

It will turn out that $\psi_r(x)$ also satisfies a simple functional equation which relates its values at the points x and $1/x$. This result will be given as Theorem 4 in Section 6.

Let now $\xi = Re^{i\delta}$, where $R > 0$, $0 \leq \delta < \frac{\pi}{2}$, and eventually we shall let $\delta \rightarrow \frac{\pi}{2}$. Then in (2.4) we may turn the line of integration by the angle δ around the origin to obtain

$$(2.10) \quad \begin{aligned} \int_0^\infty \psi_r(x) x^{s-1} dx &= \int_0^{e^{i\delta}\infty} = \int_0^\xi + \int_\xi^{\xi\infty} \\ &= \int_{\xi^{-1}}^{\xi^{-1}\infty} \psi_r\left(\frac{1}{x}\right) x^{-s-1} dx + \int_\xi^{\xi\infty} \psi_r(x) x^{s-1} dx. \end{aligned}$$

Suppose temporarily that $\Re s > 1$. In the integral with $\psi_r(1/x)$ we use Theorem 4 (the functional equation for ψ_r) to obtain

$$(2.11) \quad \int_{\xi^{-1}}^{\xi^{-1}\infty} \psi_r\left(\frac{1}{x}\right) x^{-s-1} dx = \int_{\xi^{-1}}^{\xi^{-1}\infty} \psi_r(x) x^{-s} dx + H_r(s, \xi),$$

where the function $H_r(s, \xi)$ is defined by (2.17). It can be easily evaluated in terms of elementary functions, since

$$(2.12) \quad \int x^w \log^k x dx = \frac{d^k}{dw^k} \left(\int x^w dx \right) = \frac{d^k}{dw^k} \left(\frac{x^{w+1}}{w+1} \right)$$

for $k \in \mathbb{N}$ and $w \neq -1$. Hence applying (2.12) we obtain an analytic continuation of $H_r(s, \xi)$ which is valid for all complex s except $s = 0, 1$. For $r \geq 3$ we have, turning the line of integration so that it is again parallel to the real axis,

$$(2.13) \quad \begin{aligned} \lim_{\delta \rightarrow \frac{\pi}{2}} \int_\xi^{\xi\infty} \psi_r(x) x^{s-1} dx &= \lim_{\delta \rightarrow \frac{\pi}{2}} \int_\xi^{\xi+\infty} \psi_r(x) x^{s-1} dx \\ &= \int_{iR}^{iR+\infty} \psi_r(x) x^{s-1} dx. \end{aligned}$$

Therefore from (2.4), (2.10) and (2.11) we obtain

$$(2.14) \quad \begin{aligned} \pi^{-\frac{rs}{2}} \Gamma^r\left(\frac{s}{2}\right) \zeta^r(s) &= \int_{iR}^{iR+\infty} \psi_r(x) x^{s-1} dx \\ &+ \int_{\frac{1}{iR}}^{\frac{1}{iR}+\infty} \psi_r(x) x^{-s} dx + H_r(s, \xi). \end{aligned}$$

Hence by analytic continuation we obtain from (2.14) the desired integral representation for $\zeta^r(s)$. It generalizes the case $r = 2$ of [5], where $R = p/q$ was a rational number. The remaining details of the proof will be given in Section 7, and the result is

Theorem 1. *For $R > 0$, $s \neq 0, 1$, and $r \geq 3$ a fixed integer we have*

$$(2.15) \quad \zeta^r(s) = T_r(s, R) + X_r(s) \overline{T_r(1 - \bar{s}, R^{-1})} + \pi^{\frac{rs}{2}} \Gamma^{-r} \left(\frac{s}{2} \right) H_r(s, iR),$$

where

$$(2.16) \quad X_r(s) = \pi^{rs - \frac{r}{2}} \frac{\Gamma^r \left(\frac{1-s}{2} \right)}{\Gamma^r \left(\frac{s}{2} \right)},$$

$$T_r(s, R) = \pi^{\frac{rs}{2}} \Gamma^{-r} \left(\frac{s}{2} \right) \int_{iR}^{iR+\infty} \psi_r(x) x^{s-1} dx,$$

$$(2.17) \quad H_r(s, \xi) := \int_{\xi^{-1}}^{\xi^{-1}\infty} (xp_{r-1}(-\log x) - p_{r-1}(\log x)) x^{-s-1} dx,$$

and the polynomial $p_{r-1}(u)$ of degree $r - 1$ in u is defined by

$$p_{r-1}(\log x) = \operatorname{Res}_{s=1} \pi^{-\frac{rs}{2}} \zeta^r(s) \Gamma^r \left(\frac{s}{2} \right) x^{1-s}.$$

3. Some properties of $f_r(x)$

It is clear that the study of the function $f_r(x)$, defined by (2.1), is essential for the understanding of the properties of the crucial function $\psi_r(x)$, defined by (2.3). The function $f_r(x)$ is in fact equal to $E_{r,0}(x/2)$ in the notation of A. GUTHMANN [1], where for nonnegative integers λ and ν such that $\lambda + \nu \geq 1$, and $x, c > 0$ he defined and studied the function

$$E_{\lambda,\nu}(x) := \frac{1}{2\pi i} \int_{(c)} \Gamma^\lambda \left(\frac{s}{2} \right) \Gamma^\nu(s) x^{-s} ds,$$

with the aim of deriving approximate functional equations for a class of Dirichlet series.

We begin the discussion concerning $f_r(x)$ with the simplest case $r = 1$, when we have

$$(3.1) \quad f_1(x) = 2 \cdot \frac{1}{2\pi i} \int_{(c)} \Gamma(w) \left(\frac{x^2}{4}\right)^{-w} dw = 2e^{-\frac{x^2}{4}}.$$

Consequently (2.4) becomes

$$(3.2) \quad \begin{aligned} \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) &= \int_0^\infty \psi_1(x) x^{s-1} dx, \\ \psi_1(x) &= 2 \sum_{n=1}^\infty e^{-\pi x^2 n^2} \quad (\Re s > 1), \end{aligned}$$

and (3.2) is in fact due already to B. RIEMANN [11]. Since ψ_1 is essentially a theta-function, the functional equation for the theta-function, namely

$$(3.3) \quad \theta(z) := \sum_{n=-\infty}^\infty e^{-\pi n^2 z} = z^{-\frac{1}{2}} \sum_{n=-\infty}^\infty e^{-\pi n^2 z^{-1}} \quad (\Re z > 0),$$

may be applied to yield a classical proof of the functional equation (2.5).

For $r = 2$ we have

$$(3.4) \quad f_2(x) = \frac{1}{2\pi i} \int_{(c)} \Gamma^2\left(\frac{s}{2}\right) \left(\frac{x}{2}\right)^{-s} ds = 4K_0(x) \quad (c > 0, \Re x > 0),$$

where in standard notation

$$(3.5) \quad K_s(z) = \frac{1}{2} \int_0^\infty t^{s-1} \exp\left(-\frac{z}{2} \left(t + \frac{1}{t}\right)\right) dt \quad (\Re z > 0)$$

is the modified Bessel function of the third kind (also called Macdonald's function). One can establish the second equality in (3.4) by noting that, for $\Re s > 0$,

$$\begin{aligned} \int_0^\infty K_0(x) x^{s-1} dx &= \frac{1}{2} \int_0^\infty x^{s-1} \int_0^\infty t^{-1} \exp\left(-\frac{x}{2} \left(t + \frac{1}{t}\right)\right) dt dx \\ &= \Gamma(s) 2^{s-1} \int_0^\infty t^{-1} \left(t + \frac{1}{t}\right)^{-s} dt = \Gamma(s) 2^{s-1} \int_0^\infty \frac{t^{s-1}}{(t^2 + 1)^s} dt, \end{aligned}$$

where the change of the order of integration is justified by absolute convergence. Change of variable $x = (t^2 + 1)^{-1}$ in the last integral gives

$$(3.6) \quad \begin{aligned} \int_0^\infty K_0(x)x^{s-1}dx &= \Gamma(s)2^{s-2} \int_0^1 x^{\frac{s}{2}-1}(1-x)^{\frac{s}{2}-1}dx \\ &= \Gamma(s)2^{s-2}B\left(\frac{s}{2}, \frac{s}{2}\right) = 2^{s-2}\Gamma^2\left(\frac{s}{2}\right), \end{aligned}$$

since the beta-function $B(a, b)$ satisfies

$$B(a, b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \quad (\Re a, \Re b > 0).$$

From (3.6) one obtains (3.4) by Mellin inversion.

Also from the representation $K_0(z) = \int_1^\infty e^{-zt}(t^2 - 1)^{-1/2}dt$ ($\Re z > 0$) (see [7], p. 119), we obtain by change of variable $t = 1 + x/z$ that

$$(3.7) \quad \begin{aligned} K_0(z) &= (2z)^{-1/2}e^{-z} \left\{ \int_0^\infty e^{-x}x^{-1/2}dx \right. \\ &\quad \left. + \int_0^\infty e^{-x}x^{-1/2} \left(\left(1 + \frac{x}{2z}\right)^{-1/2} - 1 \right) dx \right\}. \end{aligned}$$

By analytic continuation (3.7) extends to the entire complex plane cut from 0 to $-\infty$. After some elementary estimations it follows from (3.7) that, in the complex z -plane cut from 0 to $-\infty$, we have

$$(3.8) \quad K_0(z) = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z}(1 + H(z)),$$

where $H(z)$ is holomorphic and the principal branch of the square root is to be taken. Moreover, for $\Re z > 0$ one has $|H(z)| \leq z$, while if additionally $|z| \geq 1$ is assumed, then $|H(z)| \leq 1/|z|$. These properties of $K_0(z)$ were used in an essential way by A. GUTHMANN [5] in his work on the integral representations for $\zeta^2(s)$.

When $r \geq 3$ the situation becomes much more complicated, since it does not seem possible to find a simple expression for $f_r(x)$ in closed form, from which one can readily deduce its analytic properties and asymptotic behaviour as $x \rightarrow \infty$. However we can obtain a series expansion for $f_r(x)$ if in the defining relation (2.1) we take $c = -N - 1/2$, where N is a natural

number tending to infinity. In doing this we pass the poles of the integrand at $s = -2n$ ($n = 0, 1, 2, \dots$), which are of order r . We have

$$\begin{aligned} \operatorname{Res}_{s=-2n} \Gamma^r \left(\frac{s}{2} \right) \left(\frac{x}{2} \right)^{-s} &= x^{2n} (a_{0,r}(n) \log^{r-1} x \\ &\quad + a_{1,r}(n) \log^{r-2} x + \dots + a_{r-1,r}(n)) \end{aligned}$$

with suitable constants $a_{j,r}(n)$ ($j = 0, 1, 2, \dots, r-1$) which may be explicitly evaluated. We recall Stirling's formula (asymptotic expansion) for $\Gamma(z)$ in the form (see N.N. LEBEDEV [7])

$$(3.9) \quad \Gamma(z) \sim e^{(z-\frac{1}{2}) \log z - z + \frac{1}{2} \log(2\pi)} \left(1 + \frac{1}{12z} + \frac{1}{288z^2} + \dots \right) \\ (z \rightarrow \infty, |\arg z| < \pi)$$

and use it to obtain that the integral along the line $\Re w = -N - \frac{1}{2}$ tends to zero as $N \rightarrow \infty$. The meaning of the symbol $\sim$ in the asymptotic expansion is, as usual, that if we stop at the n -th term in the series, then the error that is made is $O_n(|z|^{-n-1})$. Hence we obtain by the residue theorem, for $\Re x > 0$,

$$(3.10) \quad f_r(x) = \sum_{n=0}^{\infty} x^{2n} (a_{0,r}(n) \log^{r-1} x \\ + a_{1,r}(n) \log^{r-2} x + \dots + a_{r-1,r}(n)),$$

which shows that $\lim_{x \rightarrow 0+} f_r(x) = +\infty$. More precisely, since $a_{0,r}(0) = (-1)^{r-1} 2^r / (r-1)!$ we have

$$\lim_{T \rightarrow \infty} \frac{f_r\left(\frac{1}{T}\right)}{\log^{r-1} T} = \frac{2^r}{(r-1)!} \quad (r = 1, 2, \dots).$$

For $x > 0$ the function $f_r(x)$ is positive, monotonically decreasing and (3.10) gives $\lim_{x \rightarrow \infty} f_r(x) = 0$. Namely with the change of variable

$u = xt^{-1/2}$ we have, for $x > 0$,

$$\begin{aligned} 2 \int_0^\infty e^{-\frac{x^2}{u^2}} f_r(u) \frac{du}{u} &= 2 \cdot \frac{1}{2\pi i} \int_{(c)} \Gamma^r\left(\frac{s}{2}\right) 2^s \left(\int_0^\infty e^{-\frac{x^2}{u^2}} u^{-1-s} du \right) ds \\ &= \frac{1}{2\pi i} \int_{(c)} \Gamma^r\left(\frac{s}{2}\right) 2^s x^{-s} \int_0^\infty e^{-t} t^{\frac{s}{2}-1} dt ds \\ &= \frac{1}{2\pi i} \int_{(c)} \Gamma^{r+1}\left(\frac{s}{2}\right) \left(\frac{x}{2}\right)^{-s} ds = f_{r+1}(x), \end{aligned}$$

where the interchange of integration is permitted by absolute convergence. Hence for $r \geq 1$ we have

$$\begin{aligned} (3.11) \quad f_{r+1}(x) &= 2 \int_0^\infty e^{-\frac{x^2}{u^2}} f_r(u) \frac{du}{u}, \\ f'_{r+1}(x) &= -4x \int_0^\infty e^{-\frac{x^2}{u^2}} f_r(u) \frac{du}{u^3}. \end{aligned}$$

In view of (3.1) we easily conclude from (3.11) by induction that $f_r(x) > 0$, $f'_r(x) < 0$ for $x > 0$ and any $r \geq 1$.

Since the residue of $\Gamma(s)$ at $s = -n$ ($n = 0, 1, 2, \dots$) is $(-1)^n/n!$, we easily see that for $r = 1$ formula (3.10) reduces to

$$f_1(x) = 2 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{4^n n!} = 2e^{-\frac{x^2}{4}}.$$

To see explicitly the shape of (3.10) for $r = 2$ write

$$(3.12) \quad f_2(x) = \frac{2}{2\pi i} \int_{(c)} \Gamma^2(w) \left(\frac{x}{2}\right)^{-2w} dw,$$

so that now the integrand has poles of second order at $n = 0, -1, -2, \dots$. Near $w = -n$ we have the expansions

$$\begin{aligned} z^{-2w} &= z^{2n} (1 - 2(w+n) \log z + (w+n)^2 \log^2 z + \dots), \\ \Gamma(w) &= \frac{(-1)^n}{n!(w+n)} + c(n) + c_1(n)(w+n) + \dots, \\ \Gamma^2(w) &= \frac{1}{(n!)^2(w+n)^2} + \frac{2(-1)^n c(n)}{n!(w+n)} + \dots. \end{aligned}$$

Note that the constant $c(n)$ satisfies, by l'Hospital's rule,

$$\begin{aligned} c(n) &= \lim_{w \rightarrow -n} \left(\Gamma(w) - \frac{(-1)^n}{n!(w+n)} \right) = \lim_{w \rightarrow -n} \frac{(w+n)\Gamma(w) - \frac{(-1)^n}{n!}}{w+n} \\ &= \lim_{w \rightarrow -n} (\Gamma(w) + \Gamma'(w)(w+n)) = \lim_{w \rightarrow -n} (w+n)\Gamma(w) \left(\frac{1}{w+n} + \frac{\Gamma'(w)}{\Gamma(w)} \right) \\ &= \frac{(-1)^n}{n!} \lim_{w \rightarrow -n} \left(\frac{1}{w+n} + \psi(w) \right), \end{aligned}$$

where as usual

$$\psi(w) := (\log \Gamma(w))' = \frac{\Gamma'(w)}{\Gamma(w)}.$$

But from the reflection property $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$ it follows by logarithmic differentiation that

$$\psi(z) = \psi(1-z) - \pi \cot(\pi z),$$

which gives

$$c(n) = \frac{(-1)^n}{n!} \psi(n+1),$$

with (see N.N. LEBEDEV [7])

$$\psi(n+1) = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \gamma, \quad \gamma = -\Gamma'(1) = 0.57721 \dots \text{ (Euler's constant).}$$

Therefore we have

$$\operatorname{Res}_{w=-n} \Gamma^2(w) z^{-2w} = -\frac{2z^{2n}}{(n!)^2} (\log z - \psi(n+1)) \quad (n = 0, 1, 2, \dots),$$

and from (3.12) we obtain by the residue theorem

$$(3.13) \quad f_2(x) = -4 \sum_{n=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2n}}{(n!)^2} \left(\log \frac{x}{2} - \psi(n+1) \right).$$

On comparing (3.4) and (3.13) we obtain

$$K_0(x) = - \sum_{n=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2n}}{(n!)^2} \left(\log \frac{x}{2} - \psi(n+1) \right),$$

which is the well-known series expansion (see N.N. LEBEDEV [7]) of $K_0(x)$.

The analysis that led to (3.10) can be carried further, as was done for the functions $E_{\lambda,\nu}(x)$ by Guthmann [1]. In particular, his Lemma 3 gives that there exist entire functions $H_{r,j}(x)$ such that, for x in the complex plane cut from $-\infty$ to 0,

$$f_r(x) = \sum_{j=0}^{r-1} H_{r,j}(x) \log^j x,$$

and the coefficients in the power series expansion for $H_{r,j}(x)$ can be evaluated explicitly.

4. The differential equation satisfied by $f_r(x)$

The function $f_r(x)$, defined by (2.1), satisfies the differential equations

$$\frac{y'}{x} = -\frac{1}{2}y \quad (r=1), \quad y'' + \frac{y'}{x} = y \quad (r=2),$$

$$xy''' + 3y'' + \frac{y'}{x} = -2y \quad (r=3),$$

$$x^2y^{(4)} + 6xy''' + 7y'' + \frac{y'}{x} = 4y \quad (r=4),$$

$$x^3y^{(5)} + 10x^2y^{(4)} + 25xy''' + 15y'' + \frac{y'}{x} = -8y \quad (r=5),$$

etc. Note that the differential equation for $r=1$ trivially follows from $f_1(x) = 2e^{-x^2/4}$, while the one for $r=2$ is a consequence of the fact that the Bessel functions $I_\nu(z)$ and $K_\nu(z)$ are (linearly independent) solutions of the differential equation

$$y'' + \frac{y'}{x} - \left(1 + \frac{\nu^2}{x^2}\right)y = 0.$$

In general, $f_r(x)$ satisfies a relatively simple differential equation of order r , which is linear and homogeneous, and whose special cases are the examples given above. This fact may provide useful information on the behaviour of $f_r(x)$. Recall that the Stirling numbers $S(n, m)$ of the second kind denote

the number of ways of partitioning a set of $n (\geq m) \geq 1$ elements into m non-empty subsets. They satisfy the relation

$$(4.1) \quad x^n = \sum_{m=1}^n S(n, m)x(x-1)\dots(x-m+1).$$

Then we have

Theorem 2. *For $r \geq 1$ the function $y = f_r(x)$ satisfies the differential equation*

$$(4.2) \quad \sum_{j=1}^r S(r, j)x^{j-2}y^{(j)} = (-1)^r 2^{r-2}y.$$

PROOF. Setting $x = -s$ in (4.1) we obtain

$$(4.3) \quad (-1)^n s^n = \sum_{m=1}^n (-1)^m S(n, m)s(s+1)\dots(s+m-1).$$

From (2.1) we obtain

$$y^{(j)} = f_r^{(j)}(x) = \frac{(-1)^j}{2\pi i} \int_{(c)} \Gamma^r\left(\frac{s}{2}\right) 2^s s(s+1)\dots(s+j-1)x^{-s-j} ds$$

$$(c > 0, j = 1, 2, \dots).$$

In view of $z\Gamma(z) = \Gamma(z+1)$ we obtain from (4.3) (with $n = r$), for $c > 2$,

$$\begin{aligned} \sum_{j=1}^r S(r, j)x^{j-2}y^{(j)} &= \frac{1}{2\pi i} \int_{(c)} \Gamma^r\left(\frac{s}{2}\right) 2^s x^{-s-2} \\ &\quad \times \sum_{j=1}^r (-1)^j S(r, j)s(s+1)\dots(s+j-1) ds \\ &= \frac{(-1)^r}{2\pi i} \int_{(c)} \Gamma^r\left(\frac{s}{2}\right) 2^s s^r x^{-s-2} ds \\ &= \frac{(-1)^r}{2\pi i} 2^{r-2} \cdot \int_{(c)} \Gamma^r\left(\frac{s+2}{2}\right) 2^{s+2} x^{-(s+2)} ds \\ &= (-1)^r 2^{r-2} \cdot \frac{1}{2\pi i} \int_{(c-2)} \Gamma^r\left(\frac{w}{2}\right) \left(\frac{x}{2}\right)^{-w} dw = (-1)^r 2^{r-2}y. \end{aligned}$$

Since $S(r, 1) = S(r, r) = 1$, $S(r, r-1) = \frac{r}{2}(r-1)$ we easily compute $S(r, j)$ for $1 \leq r, j \leq 5$ to obtain from (4.2) the examples stated at the beginning of this section. Further examples can be obtained by using a table of Stirling numbers of the second kind.

One can obtain another variant of the differential equation satisfied by $f_r(x)$. Namely, let

$$(4.4) \quad \varphi_r(x) := f_r(2\sqrt{x}),$$

so that

$$(4.5) \quad f_r(x) = \varphi_r\left(\frac{x^2}{4}\right).$$

Then we obtain

$$(4.6) \quad \varphi_r(x) = \frac{2}{2\pi i} \int_{(c)} \Gamma^r(w) x^{-w} dw \quad (c > 0, \Re, x > 0).$$

If Δ is the differential operator defined by

$$\Delta^1 f(x) = \Delta f(x) := x f'(x), \quad \Delta^r = \Delta(\Delta^{r-1}) \quad (r \geq 2),$$

then from (4.6) and the functional equation $z\Gamma(z) = \Gamma(z+1)$ we obtain the differential equation satisfied by $\varphi_r(x)$ in the form

$$(4.7) \quad \Delta^r \varphi_r(x) = (-1)^r x \varphi_r(x).$$

Similarly to (4.2), we can ascertain that (4.7) is a also linear, homogeneous differential equation of order r . The equation (4.7) is in fact equivalent to equation (42) of A. Guthmann [1], but the advantage of (4.2) over (4.7) is that (4.2) gives quite explicitly the shape of the differential equation in question, whereas (4.7) does not.

5. The asymptotic expansion of $f_r(x)$

It seems of interest to find the asymptotic expansion of $f_r(x)$ and $\psi_r(x)$ as $x \rightarrow \infty$ in terms of elementary functions. Consider first $f_r(x)$.

For $r = 1$ there is nothing to be done since $f_1(x) = 2e^{-x^2/4}$, and for $r = 2$ we have

$$(5.1) \quad f_2(x) \sim \left(\frac{8\pi}{x}\right)^{1/2} e^{-x} \left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n [(2n-1)!!]^2}{2^{3n} n!} x^{-n}\right) \\ (\Re x > 0).$$

The asymptotic expansion (5.1) follows from (3.4) and the corresponding asymptotic expansion (see (5.11.9) of N.N. LEBEDEV [7])

$$K_\nu(z) \sim \left(\frac{\pi}{2z}\right)^{1/2} e^{-z} \sum_{n=0}^{\infty} (\nu, n) (2z)^{-n} \quad (|\arg z| \leq \pi - \delta),$$

where $0 < \delta < \frac{\pi}{2}$, $(\nu, 0) = 1$ and for $k \geq 1$

$$(\nu, k) = \frac{(4\nu^2 - 1^2)(4\nu^2 - 3^2) \dots (4\nu^2 - (2k-1)^2)}{2^{2k} k!}.$$

For the general case of $f_r(x)$ we could appeal to Lemma 6 of A. GUTHMANN [1], who gave the asymptotic expansion of the function

$$E_{\lambda, \nu}(x) = \frac{1}{2\pi i} \int_{(c)} \Gamma^\lambda \left(\frac{s}{2}\right) \Gamma^\nu(s) x^{-s} ds \quad (c > 0),$$

and use the fact that $f_r(x) = E_{r,0}(x/2)$. Guthmann's proof of the asymptotic expansion of $E_{\lambda, \nu}(x)$ is long and complicated. It uses, among other things, the convolution property that

$$\int_0^\infty f(u) g\left(\frac{x}{u}\right) \frac{du}{u} = \frac{1}{2\pi i} \int_{(c)} F(s) G(s) x^{-s} ds,$$

if f, F and g, G are two pairs of Mellin transforms, and certain conditions are satisfied. We shall obtain here the asymptotic expansion of $f_r(x)$ by another method, which is simpler and can be readily generalized. Namely if $a_1, a_2, \dots, a_r > 0$, let us define

$$(5.2) \quad H(x) = H(x; a_1, \dots, a_r) \\ = \frac{1}{2\pi i} \int_{(c)} \Gamma(a_1 w) \Gamma(a_2 w) \dots \Gamma(a_r w) x^{-w} dw \quad (c > 0).$$

Then for $a_1 = \dots = a_r = 1/2$ we obtain $H(x) = f_r(2x)$, and for $a_1 = \dots = a_\lambda = 1/2, a_{\lambda+1} = \dots = a_r = 1$ ($r = \lambda + \nu$) we obtain $H(x) = E_{\lambda,\nu}(x)$. We have

Theorem 3. *Let $x \rightarrow \infty$ and*

$$(5.3) \quad \begin{aligned} B &= \frac{(2\pi)^{\frac{r-1}{2}}}{\sqrt{a_1 \cdots a_r (a_1 + \cdots + a_r)}} \left(a_1^{a_1} \cdots a_r^{a_r} \right)^{\frac{r-1}{2(a_1 + \cdots + a_r)}}, \\ D &= \frac{a_1 + \cdots + a_r}{\left(a_1^{a_1} \cdots a_r^{a_r} \right)^{\frac{1}{a_1 + \cdots + a_r}}}, \quad X = Dx^{\frac{1}{a_1 + \cdots + a_r}}. \end{aligned}$$

Then there exist constants $e_1, e_2, \dots$, which can be explicitly evaluated and which depend on $a_1, \dots, a_r$, such that

$$(5.4) \quad H(x) \sim Bx^{\frac{1-r}{2(a_1 + \cdots + a_r)}} e^{-X} \left(1 + \frac{e_1}{X} + \frac{e_2}{X^2} + \cdots \right).$$

PROOF. The basic tool in the proof is Stirling's formula (3.9), which for $a > 0$ gives, with $|\arg z| < \pi$ and some constants $c_1, c_2, \dots$ depending on a and b ,

$$(5.5) \quad \Gamma(az + b) \sim \sqrt{2\pi} e^{-az} (az)^{az+b-1/2} \left(1 + \frac{c_1}{z} + \frac{c_2}{z^2} + \cdots \right).$$

Using (5.5) we obtain, for some constants $d_1, d_2, \dots$ which depend on $a_1, \dots, a_r$,

$$(5.6) \quad \begin{aligned} &\frac{\Gamma(a_1 w) \cdots \Gamma(a_r w)}{\Gamma\left((a_1 + \cdots + a_r)w + \frac{1-r}{2}\right)} \sim \left(1 + \frac{d_1}{w} + \frac{d_2}{w^2} + \cdots \right) \\ &\times (2\pi)^{\frac{r-1}{2}} \exp \left\{ w \left(a_1 \log a_1 + \cdots + a_r \log a_r \right. \right. \\ &\quad \left. \left. - (a_1 + \cdots + a_r) \log(a_1 + \cdots + a_r) \right) \right. \\ &\quad \left. - \frac{1}{2} \log a_1 - \cdots - \frac{1}{2} \log a_r + \frac{r}{2} \log(a_1 + \cdots + a_r) \right\}. \end{aligned}$$

The transformation formula (5.6) is the crucial step in deriving (5.4). Now we take $c(a_1 + \cdots + a_r) > N + 1$ in (5.2), where $N \geq 1$ is any fixed integer, insert (5.6) and make the change of variable $(a_1 + \cdots + a_r)w + \frac{1-r}{2} = s$. If B and D are given by (5.3), we obtain, for a suitable constant $C > N + 1$,

suitable constants e_j (which depend on $a_1, a_2, \dots$ and may be evaluated explicitly) and a function $h_N(s)$ which is regular and $\ll 1$ for $\Re s \geq N+1$,

$$(5.7) \quad \begin{aligned} H(x) = & \frac{B}{2\pi i} \int_{(c)} \Gamma(s) D^{-s} x^{\frac{1-r-2s}{2(a_1+\dots+a_r)}} \\ & \times \left(1 + \frac{e_1}{s-1} + \dots + \frac{e_N}{(s-1)(s-2)\dots(s-N)} \right. \\ & \left. + \frac{h_N(s)}{(s-1)(s-2)\dots(s-N-1)} \right) ds. \end{aligned}$$

If we use

$$z\Gamma(z) = \Gamma(z+1), \quad e^{-x} = \frac{1}{2\pi i} \int_{(c)} \Gamma(s) x^{-s} ds \quad (c > 0, \Re x > 0),$$

then we obtain from (5.7)

$$(5.8) \quad \begin{aligned} H(x) = & Bx^{\frac{1-r}{2(a_1+\dots+a_r)}} \left\{ e^{-X} \left(1 + \frac{e_1}{X} + \dots + \frac{e_N}{X^N} \right) \right. \\ & \left. + O_N \left(\left| \int_{(C)} \Gamma(s-N-1) h_N(s) X^{-s} ds \right| \right) \right\} \quad (C > N+1), \end{aligned}$$

where $X = Dx^{1/(a_1+\dots+a_r)}$. Hence (5.4) will follow if we can show that, for $C > N+1$ and $Y \rightarrow \infty$,

$$(5.9) \quad I_C(Y) := \frac{1}{2\pi i} \int_{(C)} \Gamma(s-N-1) h_N(s) Y^{-s} ds \ll Y^{-N-1} e^{-Y}.$$

To obtain (5.9) let $s = N+1+w$ and use the duplication formula for $\Gamma(s)$ in the form

$$\Gamma(s) = \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s}{2} + \frac{1}{2}\right) 2^{s-1} \pi^{-1/2}.$$

Then by the Cauchy-Schwarz inequality for integrals and (2.7)–(2.8) we

obtain (changing $C - N - 1$ to C)

$$\begin{aligned}
 I_C(Y) &= \frac{Y^{-N-1}}{2\pi i} \int_{(C)} \Gamma(w) h_N(N+1+w) Y^{-w} dw \\
 &\ll Y^{-N-1-C} \int_0^\infty |\Gamma(C+iv)| dv \\
 &\ll 2^C Y^{-N-1-C} \int_0^\infty \left| \Gamma\left(\frac{C}{2} + \frac{iv}{2}\right) \Gamma\left(\frac{C+1}{2} + \frac{iv}{2}\right) \right| dv \\
 &\ll 2^C Y^{-N-1-C} \left(\int_0^\infty \left| \Gamma\left(\frac{C}{2} + \frac{iv}{2}\right) \right|^2 dv \right. \\
 &\quad \times \left. \int_0^\infty \left| \Gamma\left(\frac{C+1}{2} + \frac{iy}{2}\right) \right|^2 dy \right)^{1/2} \\
 &\ll \left(\Gamma(C) \Gamma(C+1) \right)^{1/2} Y^{-N-1-C} = C^{1/2} \Gamma(C) Y^{-N-1-C} \\
 &\ll e^{C \log C - C - C \log Y} Y^{-N-1} \ll e^{-Y} Y^{-N-1}
 \end{aligned}$$

with the choice $C = Y$. Thus (5.4) follows from (5.8) and (5.9), since N may be arbitrary.

From (5.3) we obtain the asymptotic expansion of $f_r(x) = H(x/2)$ for $a_1 = \dots = a_r = 1$. In this case we can calculate explicitly without much trouble the first few coefficients e_j . The result is

Corollary 1. *If $r \geq 1$, $\Re x > 0$ and $x \rightarrow \infty$, then we have*

$$\begin{aligned}
 (5.10) \quad f_r(x) &\sim 2r^{-\frac{1}{2}} (2\pi)^{\frac{r-1}{2}} 2^{\frac{r-1}{r}} x^{\frac{1-r}{r}} e^{-r(\frac{x}{2})^{\frac{2}{r}}} \\
 &\quad \times \left(c_{0,r} + c_{1,r} x^{-\frac{2}{r}} + c_{2,r} x^{-\frac{4}{r}} + \dots \right),
 \end{aligned}$$

where $c_{0,r} = 1$, $c_{1,r} = 2^{2/r}(1-r^2)/(24r)$, and the other $c_{j,r}$'s are also effectively computable. Note that the range of validity of the asymptotic expansion in Theorem 3 can be extended to $|\arg x| < \frac{\pi}{2}(a_1 + \dots + a_r)$ and in Corollary 1 to $|\arg x| < \frac{\pi r}{4}$.

Setting $r = 2$ in (5.10) we obtain

$$f_2(x) = \left(\frac{8\pi}{x} \right)^{1/2} e^{-x} \left(1 - \frac{1}{8x} + O\left(\frac{1}{|x|^2}\right) \right),$$

which is in accordance with (5.1). Also we may note that by the method of proof of (5.10) we may similarly obtain an asymptotic expansion of $\psi_r(x)$, starting from (2.3) and noting that

$$\zeta^r(s) = 1 + O_r\left(\frac{1}{2^\sigma}\right) \quad (\sigma = \Re s \rightarrow \infty).$$

We shall obtain

Corollary 2. *For $r \geq 1$, $\Re x > 0$ and $x \rightarrow \infty$*

$$(5.11) \quad \psi_r(x) \sim x^{\frac{1-r}{r}} e^{-\pi r x^{\frac{2}{r}}} \left(b_{0,r} + b_{1,r} x^{-\frac{2}{r}} + b_{2,r} x^{-\frac{4}{r}} + \dots \right)$$

with suitable coefficients $b_{j,r}$ ($j = 0, 1, 2, \dots$), which may be explicitly evaluated.

If in (2.1) we take $0 < c < 1$, replace s by w then by absolute convergence for $\Re s > 0$ we may change the order of integration to obtain

$$(5.12) \quad \begin{aligned} \int_0^\infty f_r(x) e^{-sx} dx &= \frac{1}{2\pi i} \int_{(c)} \Gamma^r\left(\frac{w}{2}\right) 2^w \int_0^\infty x^{-w} e^{-sx} dx dw \\ &= \frac{1}{2\pi i} \int_{(c)} \Gamma^r\left(\frac{w}{2}\right) 2^w \Gamma(1-w) s^{w-1} dw. \end{aligned}$$

Now we take $s = 1/T$, $T \rightarrow \infty$, $c = 1/2$ and make the substitution $1-w = z$. We obtain from (5.12)

$$\int_0^\infty f_r(x) e^{-x/T} dx = \frac{1}{2\pi i} \int_{(c)} \Gamma^r\left(\frac{1-z}{2}\right) 2^{1-z} \Gamma(z) T^z dz.$$

Shifting the line of integration in the last integral to $-\infty$ and applying the residue theorem we obtain

$$\int_0^\infty f_r(x) e^{-x/T} dx \sim \sum_{n=0}^\infty \Gamma^r\left(\frac{n+1}{2}\right) \frac{2^{n+1}}{n!} \cdot \left(\frac{-1}{T}\right)^n \quad (T \rightarrow \infty),$$

which is the asymptotic expansion of the Laplace transform of $f_r(x)$ as $s = 1/T \rightarrow 0+$.

Note also that the Parseval formula for Mellin transforms gives the identity

$$\int_0^\infty f_r^2(x) x^{2\sigma-1} dx = \frac{2^{1+2\sigma}}{\pi} \int_0^\infty \left| \Gamma\left(\frac{\sigma}{2} + ix\right) \right|^{2r} dx \quad (\sigma > 0),$$

which for $r = 1$ reduces to (2.7).

6. The functional equation for $\psi_r(x)$

As mentioned in Section 2, the function $\psi_r(x)$ satisfies a simple functional equation relating its values at the points x and $1/x$. This result, which is an essential ingredient in the proof of the integral representation for $\zeta^r(s)$ furnished by Theorem 1, may be obtained as follows. From (2.3) we have by the residue theorem

$$(6.1) \quad \begin{aligned} \psi_r(x) = & \operatorname{Res}_{s=1} \pi^{-\frac{rs}{2}} \zeta^r(s) \Gamma^r\left(\frac{s}{2}\right) x^{-s} \\ & + \frac{1}{2\pi i} \int_{(\frac{1}{2})} \pi^{-\frac{rs}{2}} \zeta^r(s) \Gamma^r\left(\frac{s}{2}\right) x^{-s} ds. \end{aligned}$$

Setting

$$g_r(x) := \frac{1}{2\pi i} \int_{(\frac{1}{2})} \pi^{-\frac{rs}{2}} \zeta^r(s) \Gamma^r\left(\frac{s}{2}\right) x^{-s} ds \quad (\Re x > 0)$$

we have, by the functional equation (2.5) (raised to the r -th power) and the change of variable $1 - s = w$,

$$\begin{aligned} g_r\left(\frac{1}{x}\right) &= \frac{1}{2\pi i} \int_{(\frac{1}{2})} \pi^{-\frac{rs}{2}} \zeta^r(s) \Gamma^r\left(\frac{s}{2}\right) x^s ds \\ &= \frac{1}{2\pi i} \int_{(\frac{1}{2})} \pi^{-\frac{r(1-s)}{2}} \zeta^r(1-s) \Gamma^r\left(\frac{1-s}{2}\right) x^s ds \\ &= \frac{1}{2\pi i} \int_{(\frac{1}{2})} \pi^{-\frac{rw}{2}} \zeta^r(w) \Gamma^r\left(\frac{w}{2}\right) x^{1-w} dw = x g_r(x), \end{aligned}$$

hence

$$(6.3) \quad g_r(x) = \frac{1}{x} g_r\left(\frac{1}{x}\right) \quad (\Re x > 0).$$

Since $\zeta^r(s)$ has at $s = 1$ a pole of order r , it follows that

$$(6.4) \quad \operatorname{Res}_{s=1} \pi^{-\frac{rs}{2}} \zeta^r(s) \Gamma^r\left(\frac{s}{2}\right) x^{-s} = \frac{1}{x} p_{r-1}(\log x),$$

where $p_{r-1}(u)$ is a polynomial of degree $r-1$ in u whose coefficients, which depend on r , may be effectively evaluated, since the Laurent expansions at $s = 1$ of $\zeta(s)$ and $\Gamma(s)$ are well-known. From (6.1), (6.2) and (6.4) it follows that

$$g_r(x) = \psi_r(x) - \frac{1}{x} p_{r-1}(\log x),$$

and (6.3) yields then

$$\frac{1}{x} \psi_r \left(\frac{1}{x} \right) - p_{r-1}(-\log x) = \frac{1}{x} g_r \left(\frac{1}{x} \right) = g_r(x) = \psi_r(x) - \frac{1}{x} p_{r-1}(\log x).$$

This means that we have proved

Theorem 4. *If $\psi_r(x)$ is defined by (2.3) and $p_{r-1}(\log x)$ by (6.4), then for $r \geq 1$ and $\Re x > 0$ we have*

$$(6.5) \quad \psi_r \left(\frac{1}{x} \right) = x \psi_r(x) - p_{r-1}(\log x) + x p_{r-1}(-\log x).$$

The functional equation (6.5) shows that $\psi_r(x) \asymp x^{-1} \log^{r-1} x$ as $x \rightarrow 0+$. In particular we have

$$p_0(\log x) = \operatorname{Res}_{s=1} \pi^{-\frac{s}{2}} \zeta(s) \Gamma \left(\frac{s}{2} \right) = 1,$$

since $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and $\lim_{s \rightarrow 1} \zeta(s)(s-1) = 1$. Hence

$$\psi_1 \left(\frac{1}{x} \right) = x \psi_1(x) + x - 1,$$

which also follows from (3.3), since $\psi_1(x) = \theta(x^2) - 1$. We also have

$$(6.6) \quad p_1(u) = \gamma - 2 \log 2 - \log \pi - u,$$

where γ is Euler's constant as before. Namely near $s = 1$ we have

$$\begin{aligned} \zeta^2(s) &= \frac{1}{(s-1)^2} + \frac{2\gamma}{s-1} + a + \dots, \\ (\pi x)^{-(s-1)} &= 1 - (s-1) \log(\pi x) + \dots, \\ \Gamma \left(\frac{s}{2} \right) &= \Gamma \left(\frac{1}{2} \right) + \frac{1}{2} \Gamma' \left(\frac{1}{2} \right) (s-1) + \dots. \end{aligned}$$

But in view of $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and (see N.N. LEBEDEV [7]) $\psi(\frac{1}{2}) = \frac{\Gamma'}{\Gamma}(\frac{1}{2}) = -\gamma - 2\log 2$, we have

$$\begin{aligned}\Gamma'\left(\frac{1}{2}\right) &= -\sqrt{\pi}(\gamma + 2\log 2), \\ \Gamma^2\left(\frac{s}{2}\right) &= \pi - \pi(\gamma + 2\log 2)(s-1) + \cdots.\end{aligned}$$

We obtain

$$p_1(\log x) = \frac{1}{\pi} \operatorname{Res}_{s=1} (\pi x)^{-(s-1)} \zeta^2(s) \Gamma^2\left(\frac{s}{2}\right) = \gamma - 2\log 2 - \log \pi - \log x,$$

so that (6.6) follows. By more cumbersome calculations we may evaluate p_{r-1} for $r \geq 3$.

7. Integral representations of $\zeta^r(s)$

In this section we shall complete the proof of Theorem 1, outlined in Section 2, and then furnish yet another integral representation of $\zeta^r(s)$. To complete the proof of Theorem 1 we need to show that (2.9) holds, and that (2.11)–(2.13) follows from (2.10). To prove (2.9) it is enough to show that

$$(7.1) \quad \lim_{\delta \rightarrow \frac{\pi}{2}} \int_{\xi}^{\xi+1} \psi_r(x) x^{s-1} dx = \int_{iR}^{iR+1} \psi_r(x) x^{s-1} dx$$

$$\left(\xi = Re^{i\delta}, R > 0, r \geq 3, 0 \leq \delta < \frac{\pi}{2} \right),$$

since $\psi_r(x)$ decays exponentially at ∞ by Theorem 3. If $r \geq 3$, $\Re x > 0$, $c > 1$ is fixed, then from (2.3) and (3.9) we obtain (since $|\arg \frac{x}{2}| \leq \frac{\pi}{2}$)

$$(7.2) \quad \psi_r(x) \ll \left| \frac{x}{2} \right|^{-c} \left(1 + \int_{t_0}^{\infty} e^{t|\arg \frac{x}{2}|} t^{\frac{r(c-1)}{2}} e^{-\frac{\pi r t}{4}} dt \right) \ll |x|^{-c},$$

and a bound analogous to (7.2) will also hold for $\psi'_r(x)$. Hence from (7.2) we obtain ($\xi = Re^{i\delta}$)

$$\begin{aligned} & \int_{\xi}^{\xi+1} \psi_r(x) x^{s-1} dx - \int_{iR}^{iR+1} \psi_r(x) x^{s-1} dx \\ &= \int_0^1 \left(\psi_r(Re^{i\delta} + u)(Re^{i\delta} + u)^{s-1} - \psi_r(iR + u)(iR + u)^{s-1} \right) du \\ &= \int_0^1 \int_{Re^{\frac{i\pi}{2}}}^{Re^{i\delta}} \left(\psi'_r(v + u)(v + u)^{s-1} + \psi_r(v + u)(s-1)(v + u)^{s-2} \right) dv du \\ &\ll_{R,s} \max_{0 \leq u \leq 1} |e^{i\delta} - e^{\frac{i\pi}{2}}| \ll_{R,s} \left| \delta - \frac{\pi}{2} \right| \end{aligned}$$

since $R \leq |v + u| \leq R + 1$. Letting $\delta \rightarrow \frac{\pi}{2}$ we obtain (7.1).

To complete the proof of Theorem 1, note that (2.10) gives

$$(7.3) \quad \begin{aligned} \zeta^r(s) &= T_r(s, R) \\ &+ \pi^{\frac{rs}{2}} \Gamma^{-r} \left(\frac{s}{2} \right) \int_{\frac{1}{iR}}^{\frac{1}{iR} + \infty} \psi_r(x) x^{-s} dx + \pi^{\frac{rs}{2}} \Gamma^{-r} \left(\frac{s}{2} \right) H_r(s, iR), \end{aligned}$$

where $T_r(s, R)$ is given by (2.16) and $H_r(s, iR)$ by (2.17). Since $\overline{\psi_r(x)} = \psi_r(\bar{x})$ and $\overline{a^b} = \bar{a}^{\bar{b}}$, we obtain

$$\begin{aligned} \overline{T_r(1 - \bar{s}, R^{-1})} &= \pi^{\frac{r(1-s)}{2}} \Gamma^{-r} \left(\frac{1-s}{2} \right) \overline{\int_{\frac{1}{iR}}^{\frac{1}{iR} + \infty} \psi_r(x) x^{-\bar{s}} dx} \\ &= \pi^{\frac{r(1-s)}{2}} \Gamma^{-r} \left(\frac{1-s}{2} \right) \overline{\int_0^\infty \psi_r \left(\frac{i}{R} + u \right) \left(\frac{i}{R} + u \right)^{-\bar{s}} du} \\ &= \pi^{\frac{r(1-s)}{2}} \Gamma^{-r} \left(\frac{1-s}{2} \right) \int_0^\infty \psi_r \left(-\frac{i}{R} + u \right) \left(-\frac{i}{R} + u \right)^{-s} du \\ &= \pi^{\frac{r(1-s)}{2}} \Gamma^{-r} \left(\frac{1-s}{2} \right) \int_0^\infty \psi_r \left(\frac{1}{iR} + u \right) \left(\frac{1}{iR} + u \right)^{-s} du \\ &= \pi^{\frac{r(1-s)}{2}} \Gamma^{-r} \left(\frac{1-s}{2} \right) \int_{\frac{1}{iR}}^{\frac{1}{iR} + \infty} \psi_r(x) x^{-s} dx. \end{aligned}$$

Therefore

$$\int_{\frac{1}{iR}}^{\frac{1}{iR}+\infty} \psi_r(x) x^{-s} dx = \pi^{\frac{r(s-1)}{2}} \Gamma^r \left(\frac{1-s}{2} \right) \overline{T_r(1-\bar{s}, R^{-1})},$$

and if we insert this expression in (7.3) we obtain (2.15). Although the value of R is not specified, from the definition of T_r it follows that optimal symmetry in (2.15) will be attained in the case when $R = 1$.

Another integral representation for $T_r(s, R)$, which generalizes equation (3.1) of A. GUTHMANN [5], is given by

Theorem 5. *If $T_r(s, R)$ is given by (2.16) and $0 \leq \varphi \leq \frac{\pi}{2}$, then for $r \geq 1$ fixed we have*

$$(7.4) \quad T_r(s, R) = \sin^r \left(\frac{\pi s}{2} \right) \int_0^{\infty e^{-i\varphi}} u^{1-s} \left(\int_{iR}^{iR+\infty} x \psi_r(x) f_r(2\pi^{\frac{r}{2}} x u) dx \right) du.$$

PROOF. The result is formulated for $r \geq 1$, since it does not depend on (7.2), like the proof of (2.13) does. For $\Re s < 2$ we have, from (2.2),

$$(7.5) \quad \Gamma^r \left(\frac{2-s}{2} \right) 2^{2-s} = \int_0^\infty f_r(x) x^{1-s} dx.$$

From $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$ with $z = (2-s)/2$ we have

$$\Gamma^r \left(\frac{2-s}{2} \right) = \frac{\pi^r}{\sin^r \left(\frac{\pi s}{2} \right) \Gamma^r \left(\frac{s}{2} \right)}.$$

Hence (7.5) yields

$$(7.6) \quad \frac{1}{\Gamma^r \left(\frac{s}{2} \right)} = 2^{s-2} \pi^{-r} \sin^r \left(\frac{\pi s}{2} \right) \int_0^\infty f_r(z) z^{1-s} dz.$$

In (7.6) we turn the line of integration about the origin by the angle φ and make the change of variable $z = \alpha u$ to obtain

$$(7.7) \quad \frac{1}{\Gamma^r \left(\frac{s}{2} \right)} = 2^{s-2} \pi^{-r} \sin^r \left(\frac{\pi s}{2} \right) \alpha^{2-s} \int_0^{\infty e^{-i\varphi}} f_r(\alpha u) u^{1-s} du.$$

Taking $\alpha = 2\pi^{\frac{r}{2}}x$ we obtain then from (7.7)

$$\frac{\pi^{\frac{rs}{2}}}{\Gamma^r\left(\frac{s}{2}\right)} = \sin^r\left(\frac{\pi s}{2}\right) x^{2-s} \int_0^{\infty e^{-i\varphi}} f_r(2\pi^{\frac{r}{2}}xu) u^{1-s} du.$$

From this expression and the definition (2.16) of $T_r(s, R)$ we have

$$(7.8) \quad T_r(s, R) = \sin^r\left(\frac{\pi s}{2}\right) \int_{iR}^{iR+\infty} x \psi_r(x) \int_0^{\infty e^{-i\varphi}} u^{1-s} f_r(2\pi^{\frac{r}{2}}xu) du dx.$$

By absolute convergence of the double integral we may change the order of integration in (7.8), and (7.4) follows.

The integral representations for $\zeta^r(s)$ furnished by Theorem 1 and Theorem 5 may be regarded as initial steps towards an asymptotic expansion of $\zeta^r(s)$ (in terms of elementary functions). Namely from (1.1) it is possible to derive an asymptotic expansion of $\zeta(s)$, as was shown by C.L. SIEGEL [12]. A variant of this important formula states that

$$(7.9) \quad \begin{aligned} \zeta(s) &= \sum_{n=1}^N n^{-s} + \chi(s) \sum_{n=1}^N n^{s-1} \\ &\quad + (-1)^{N-1} (2\pi)^{\frac{s+1}{2}} \Gamma^{-1}(s) t^{\frac{s-1}{2}} e^{\pi i s - \frac{it}{2} - \frac{\pi i}{8}} S, \end{aligned}$$

where $0 \leq \Re s \leq 1$, $t = \Im s \rightarrow \infty$, $N = \left\lfloor \sqrt{\frac{t}{2\pi}} \right\rfloor$, $\chi(s) = \zeta(s)/\zeta(1-s)$,

$$S = \sum_{k=0}^{\nu-1} a_k \sum_{0 \leq 2r \leq k} b_{kr} F^{(k-2r)}(\delta) + O((n/t)^{\nu/6}), \quad \delta = \sqrt{t} - \left(N + \frac{1}{2}\right) \sqrt{2\pi},$$

a_k, b_{kr} are certain complex constants with $a_k \ll t^{-k/6}$, $\nu (\leq 2 \cdot 10^{-8}t)$ is a natural number, and

$$F(z) := \frac{\cos(z^2 + \frac{3\pi}{8})}{\cos(\sqrt{2\pi}z)}.$$

The integral representations for $\zeta^2(s)$ of A. GUTHMANN [5] have not yielded yet an asymptotic expansion for $\zeta^2(s)$ of the desired form

$$(7.10) \quad \zeta^2(s) = \sum_{n \leq \frac{t}{2\pi}} d(n) n^{-s} + \chi^2(s) \sum_{n \leq \frac{t}{2\pi}} d(n) n^{s-1} + R\left(s, \frac{t}{2\pi}\right)$$

with an explicit expression for $R(s, \frac{t}{2\pi})$ in terms of elementary functions, of the type obtained by Y. MOTOHASHI [8]–[10]. The latter involve the function $\Delta(x)$, which represents the error term in the Dirichlet divisor problem and a related function. In the general case one would like to obtain

$$(7.11) \quad \zeta^r(s) = \sum_{n \leq (\frac{t}{2\pi})^{r/2}} d_r(n) n^{-s} + \chi^r(s) \sum_{n \leq (\frac{t}{2\pi})^{r/2}} d_r(n) n^{s-1} + R_r(s, \frac{t}{2\pi}),$$

where $R_r(s, \frac{t}{2\pi})$ is to be considered as the error term in the approximate functional equation (7.11). In the most important case $s = \frac{1}{2} + it$ some results on $R_r(s, \frac{t}{2\pi})$ are to be found in Ch. 4 of the author's monograph [6]. It would be certainly interesting if one could use the integral representations furnished by Theorem 1 or Theorem 5 to improve the bounds for $R_r(s, \frac{t}{2\pi})$ given in [6].

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