

Some properties of the quadrinomials

$p(z) = 1 + \kappa(z + z^{N-1}) + z^N$ and $q(z) = 1 + \kappa(z - z^{N-1}) - z^N$

By DMITRIY DMITRISHIN (Odesa) and ALEXANDER STOKOLOS (Statesboro)

Abstract. We show that all the zeros of the quadrinomial $p(z) = 1 + \kappa(z + z^{N-1}) + z^N$ lie on the unit circle if and only if the inequalities

$$-1 \leq \kappa \leq \begin{cases} 1, & \text{if } N \text{ is even,} \\ N/(N-2), & \text{if } N \text{ is odd} \end{cases}$$

hold. For the quadrinomial $q(z) = 1 + \kappa(z - z^{N-1}) - z^N$, the corresponding inequalities are

$$-N/(N-2) \leq \kappa \leq \begin{cases} 1, & \text{if } N \text{ is odd,} \\ N/(N-2), & \text{if } N \text{ is even.} \end{cases}$$

In the cases of limiting values of the parameter κ , we provide factorization formulas for the corresponding quadrinomials. For example, when N is *odd* and $\kappa = N/(N-2)$, the following representation is valid: $p(z) = (1+z)^3 \prod_{j=1}^{(N-3)/2} [1+z^2-2z\gamma_j]$, where $\gamma_j = 1 - 2\nu_j^2$ with $\{\nu_j\}_{j=1}^{(N-3)/2}$ being the collection of positive roots of the equation $U'_{N-2}(x) = 0$; here $U_j(x) = U_j(\cos t) = \frac{\sin(j+1)t}{\sin t} = 2^j x^j + \dots$ are Chebyshev polynomials of the second kind, and $U'_j(x)$ are their derivatives. Similar factorization formulas are also provided for $q(z)$. As an application of the obtained results, we give the factorization formulas for the derivative of the Fejér polynomial, as well as construct certain univalent polynomials related to the polynomials $p(z)$ and $q(z)$.

DMITRIY DMITRISHIN
ODESSA NATIONAL POLYTECHNIC
UNIVERSITY
1 SHEVCHENKO AVE.
ODESA 65044
UKRAINE

ALEXANDER STOKOLOS
GEORGIA SOUTHERN UNIVERSITY
STATESBORO GA, 30460
USA

Mathematics Subject Classification: 26C10.

Key words and phrases: polynomials with zeros on the unit disk, Chebyshev polynomials of the second kind, derivative of the Chebyshev polynomial of the second kind, univalent polynomials.