

A limit theorem in the theory of finite Abelian groups

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In honour of Professors Zoltán Daróczy and Imre Káta
on their 60th birthday

Abstract. In the paper a limit theorem in the sense of the weak convergence of probability measures for a Dirichlet series used in the theory of finite Abelian groups in the space of analytic functions is obtained.

1. Introduction

Let $\mathcal{G}$ be a finite Abelian group of order $|\mathcal{G}|$. Denote by $\tau(\mathcal{G})$ and $r(\mathcal{G})$ the number of subgroups of $\mathcal{G}$ and the rank of $\mathcal{G}$, respectively. Let, as usual, $\mathbb{R}$, $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{C}$ denote the sets of real, natural, integer and complex numbers, respectively. It is known that the group $\mathcal{G}$ has rank r if

$$\mathcal{G} \cong \mathbb{Z}/m_1\mathbb{Z} \otimes \cdots \otimes \mathbb{Z}/m_r\mathbb{Z},$$

where $m_j \mid m_{j+1}$ for $j = 1, \dots, r-1$.

Let

$$t_r(m) = \sum_{|\mathcal{G}|=m, r(\mathcal{G}) \leq r} \tau(\mathcal{G}).$$

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In several recent papers the sum function

$$T(x) = \sum_{m \leq x} t_2(m)$$

has been studied. Let

$$\Delta(x) = T(x) - K_1 x \log^2 x - K_2 x \log x - K_3 x,$$

where K_1 , K_2 and K_3 are effective constants. Denote by B a number bounded by a constant. G. BHOWMIK and H. MENZER [3] proved that

$$(1) \quad \Delta(x) = Bx^{c+\varepsilon}$$

with $c = 31/43$ for every positive ε . H. MENZER [12] improved the estimate (1) until $c = 9/14$, and he also conjectured that

$$\Delta(x) = \Omega(x^{1/2} \log^2 x).$$

The latter conjecture was proved by G. BHOWMIK and J. WU in [4]. Moreover, they obtained a bound

$$\Delta(x) = Bx^{5/8} \log^4 x.$$

Finally, A. IVIČ [6] investigated the mean square of the error term $\Delta(x)$ and proved that

$$\int_1^x \Delta^2(u) du = Bx^2 (\log x)^{31/3} (\log \log x)^{28/3},$$

and

$$\int_1^x \Delta^2(u) du = \Omega(x^2 \log^4 x).$$

These results allowed him to conjecture that

$$\int_1^x \Delta^2(u) du \sim Cx^2 \log^4 x, \quad x \rightarrow \infty,$$

with a suitable constant $C > 0$.

Let $s = \sigma + it$ be a complex variable. The Dirichlet series

$$H(s) = \sum_{m=1}^{\infty} \frac{t_2(m)}{m^s}, \quad \sigma > 1,$$

plays an important role in the proofs of the results mentioned above. G. BHOWMIK and O. RAMARÉ [2], see also [4], obtained that, for $\sigma > 1/2$,

$$(2) \quad H(s) = \zeta^2(s)\zeta^2(2s)\zeta(2s-1) \prod_p \left(1 + \frac{1}{p^{2s}} - \frac{2}{p^{3s}}\right),$$

where, as usual, $\zeta(s)$ stands for the Riemann zeta-function. This gives the analytic continuation of $H(s)$ over the half-plane $\sigma > 1/2$ except for the pole of order 3 at the point $s = 1$. Our aim is to study the statistical properties of the function $H(s)$, and in this note we will prove a limit theorem in the sense of the weak convergence of probability measures in the space of analytic functions for $H(s)$. The theory of functional limit theorems for Dirichlet series was obtained by B. BAGCHI in [1] (see also [8]), and we will use his ideas.

Let γ be the unit circle on the complex plane $\mathbb{C}$, i.e. $\gamma = \{s \in \mathbb{C} : |s|=1\}$, and let

$$\Omega = \prod_p \gamma_p,$$

where $\gamma_p = \gamma$ for each prime number p . With the product topology and pointwise multiplication Ω is a compact Abelian topological group. Denote by $\mathcal{B}(S)$ the class of Borel sets of the space S . Then there exists the probability Haar measure m_H on $(\Omega, \mathcal{B}(\Omega))$. This yields a probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Let $\omega(p)$ stand for the projection of $\omega \in \Omega$ to the coordinate space γ_p . Setting

$$\omega(k) = \prod_{p^\alpha \parallel k} \omega^\alpha(p),$$

where $p^\alpha \parallel k$ means that $p^\alpha \mid k$ but $p^{\alpha+1} \nmid k$, we obtain an extension of $\omega(p)$ to the set $\mathbb{N}$ as a completely multiplicative unimodular function.

Let G be a region on $\mathbb{C}$. Denote by $H(G)$ and $M(G)$ the spaces of analytic and meromorphic functions on G , respectively, equipped with the topology of uniform convergence on compacta. Let $D = \{s \in \mathbb{C} : \sigma > 3/4\}$. Now we may define an $H(D)$ -valued random element on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$:

$$\begin{aligned} H(s, \omega) &= \prod_p \left(1 - \frac{\omega(p)}{p^s}\right)^{-2} \prod_p \left(1 - \frac{\omega^2(p)}{p^{2s}}\right)^{-2} \prod_p \left(1 - \frac{\omega^2(p)}{p^{2s-1}}\right)^{-1} \\ &\quad \times \prod_p \left(1 + \frac{\omega^2(p)}{p^{2s}} - \frac{2\omega^3(p)}{p^{3s}}\right), \quad \omega \in \Omega, \quad s \in D. \end{aligned}$$

The second and the last products in this formula converge uniformly for $\sigma > 3/4$, and therefore, they define $H(D)$ -valued random elements. It is easy to check similarly as in [8] that the first and the third products converge uniformly for almost all $\omega \in \Omega$ on every compact subsets of D . Hence they define $H(D)$ -valued random elements. Thus $H(s, \omega)$ is an $H(D)$ -valued random element defined on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$.

Let $w(t)$ be a positive function of bounded variation on $[T_0, \infty)$, $T_0 > 0$, such that its variation $V_a^b w$ on $[a, b]$ satisfies the inequality $V_a^b w \leq cw(b) - cw(a)$ with some $c > 0$ for all $b \geq a \geq T_0$. Moreover, let

$$U = U(T, w) = \int_{T_0}^T w(t) dt,$$

and suppose that $\lim_{T \rightarrow \infty} U(T, w) = \infty$. In addition we assume that the function $w(t)$ satisfies some special condition related to the ergodic theory. Let $X(\tau, \omega)$ be an ergodic process with $E|X(\tau, \omega)| < \infty$, and let its sample paths be integrable almost surely in the Riemann sense over every finite interval. Here EX denotes the mean of the random variable X . Then we suppose that

$$(3) \quad \frac{1}{U} \int_{T_0}^T w(\tau) X(t + \tau, \omega) d\tau = EX(0, \omega) + o(1 + |t|)^\alpha$$

almost surely for all $t \in \mathbb{R}$ with some $\alpha > 0$ as $T \rightarrow \infty$. The latter equality is a generalization of the classical Birkhoff–Khinchin theorem which asserts that

$$(4) \quad \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T X(\tau, \omega) d\tau = EX(0, \omega)$$

almost surely. Thus (3) with $w(t) \equiv 1$ implies (4). Some examples of functions satisfying (3) can be found in [11].

Let $D_1 = \{s \in \mathbb{C} : 3/4 < \sigma < 1\}$, $D_2 = \{s \in \mathbb{C} : \sigma > 1\}$, and let I_A denote the indicator function of the set A . Define two probability measures

$$P_{j,T,w}(A) = \frac{1}{U} \int_{T_0}^T w(\tau) I_{\{\tau: H(s+i\tau) \in A\}} d\tau, \quad A \in \mathcal{B}(H(D_j)), \quad j = 1, 2.$$

Let P_ξ denote the distribution of an $H(D)$ -valued random element ξ , and let $P_{j,\xi}$ be the restriction of P_ξ to $(H(D_j), \mathcal{B}(H(D_j)))$, $j = 1, 2$.

Theorem. *The measures $P_{j,T,w}$ converge weakly to $P_{j,H}$ as $T \rightarrow \infty$.*

Denote by $\text{meas}\{A\}$ the Lebesgue measure of the set A , and set, for $T > 0$,

$$\nu_T^\tau(\cdots) = \frac{1}{T} \text{meas}\{\tau \in [0, T], \dots\},$$

where instead of dots we write a condition satisfied by τ . Now let

$$P_{j,T}(A) = \nu_T^\tau(H(s + i\tau) \in A), \quad A \in \mathcal{B}(H(D_j)), \quad j = 1, 2.$$

Corollary. *The measures $P_{j,T}$ converge weakly to $P_{j,H}$ as $T \rightarrow \infty$.*

Thus, the limit measure in the Theorem is independent of the function $w(t)$. This is a consequence of (3).

Note that the Theorem can be used for the investigation of the universality of the function $H(s)$.

Since the case $j = 2$ is simpler and similar to that of $j = 1$, we will consider the case $j = 1$ only.

2. Auxiliary results

For the proof of the theorem we will apply the fact that each multiplier in (2) has limit distribution. Let

$$Q_T^{(1)}(A) = \frac{1}{U} \int_{T_0}^T w(\tau) I_{\{\tau: \zeta(s+i\tau) \in A\}} d\tau, \quad A \in \mathcal{B}(H(D_1)),$$

and

$$\zeta_1(s, \omega) = \prod_p \left(1 - \frac{\omega(p)}{p^s}\right)^{-1}, \quad \omega \in \Omega, \quad s \in D.$$

Lemma 1. *The measure $Q_T^{(1)}$ converges weakly to P_{1,ζ_1} as $T \rightarrow \infty$.*

PROOF. The lemma is a Theorem from [11] with such a difference that in [11] $D_1 = \{s \in \mathbb{C} : 1/2 < \sigma < 1\}$.

Now let

$$Q_T^{(2)}(A) = \frac{1}{U} \int_{T_0}^T w(\tau) I_{\{\tau: \zeta(2(s+i\tau)) \in A\}} d\tau, \quad A \in \mathcal{B}(H(D_1)),$$

and

$$\zeta_2(s, \omega) = \prod_p \left(1 - \frac{\omega^2(p)}{p^{2s}} \right)^{-1}, \quad \omega \in \Omega, \quad s \in D.$$

Lemma 2. *The measure $Q_T^{(2)}$ converges weakly to P_{1, ζ_2} as $T \rightarrow \infty$.*

PROOF. It coincides with that of Lemma 1. We note only that the Dirichlet series for $\zeta(2s)$ is

$$\sum_{m=1}^{\infty} \frac{a_m}{m^s},$$

where

$$a_m = \begin{cases} 1 & \text{if } m = k^2, \\ 0 & \text{otherwise.} \end{cases}$$

Thus in this case

$$\begin{aligned} \zeta_2(s, \omega) &= \sum_{m=1}^{\infty} \frac{a_m \omega(m)}{m^s} = \sum_{m=1}^{\infty} \frac{\omega(m^2)}{m^{2s}} = \sum_{m=1}^{\infty} \frac{\omega^2(m)}{m^{2s}} \\ &= \prod_p \left(1 - \frac{\omega^2(p)}{p^{2s}} \right)^{-1}. \end{aligned}$$

Now we set

$$V_T^{(3)}(A) = \frac{1}{U} \int_{T_0}^T w(\tau) I_{\{\tau: \zeta(2(s+i\tau)-1) \in A\}} d\tau, \quad A \in \mathcal{B}(H(D_1)),$$

and

$$\xi_3(s, \omega) = \prod_p \left(1 - \frac{\omega^2(p)}{p^{2s-1}} \right)^{-1}, \quad \omega \in \Omega, \quad s \in D.$$

Lemma 3. *The measure $V_T^{(3)}$ converges weakly to P_{1, ξ_3} as $T \rightarrow \infty$.*

PROOF. It repeats the arguments of that of Lemmas 1 and 2. The Dirichlet series for $\zeta(2s-1)$ is

$$\sum_{m=1}^{\infty} \frac{a_m}{m^s},$$

where

$$a_m = \begin{cases} \sqrt{m} & \text{if } m = k^2, \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, we have

$$\begin{aligned} \xi_3(s, \omega) &= \sum_{m=1}^{\infty} \frac{a_m \omega(m)}{m^s} = \sum_{m=1}^{\infty} \frac{m \omega(m^2)}{m^{2s}} \\ &= \sum_{m=1}^{\infty} \frac{\omega^2(m)}{m^{2s-1}} = \prod_p \left(1 - \frac{\omega^2(p)}{p^{2s-1}} \right)^{-1} \end{aligned}$$

for almost all $\omega \in \Omega$.

For all $\sigma > 1/2$ let

$$U(s) = \sum_{m=1}^{\infty} \frac{u_m}{m^s} = \prod_p \left(1 + \frac{1}{p^{2s}} - \frac{2}{p^{3s}} \right),$$

the latter Dirichlet series being absolutely convergent for $\sigma > 1/2$. Moreover, let

$$V_T^{(4)}(A) = \frac{1}{U} \int_{T_0}^T w(\tau) I_{\{\tau: U(s+i\tau) \in A\}} d\tau, \quad A \in \mathcal{B}(H(D_1)),$$

and

$$\xi_4(s, \omega) = \prod_p \left(1 + \frac{\omega^2(p)}{p^{2s}} - \frac{2\omega^3(p)}{p^{3s}} \right), \quad \omega \in \Omega, \quad s \in D.$$

Lemma 4. *The measure $V_T^{(4)}$ converges weakly to P_{1, ξ_4} as $T \rightarrow \infty$.*

PROOF. We give only a sketch of the proof because of its similarity to the proof of a Theorem from [9]. Let

$$\begin{aligned} U_n(s) &= \sum_{m=1}^n \frac{u_m}{m^s}, \\ U_n(s, \omega) &= \sum_{m=1}^n \frac{u_m \omega(m)}{m^s}, \quad \omega \in \Omega. \end{aligned}$$

Then Lemma 2 of [11] asserts that the probability measures

$$\frac{1}{U} \int_{T_0}^T w(\tau) I_{\{\tau: U_n(s+i\tau) \in A\}} d\tau, \quad A \in \mathcal{B}(H(D_1)),$$

and

$$\frac{1}{U} \int_{T_0}^T w(\tau) I_{\{\tau: U_n(s+i\tau, \omega) \in A\}} d\tau, \quad A \in \mathcal{B}(H(D_1)),$$

converge weakly to the same measure as $T \rightarrow \infty$. Using this and the relations

$$\lim_{n \rightarrow \infty} \limsup_{T \rightarrow \infty} \frac{1}{U} \int_{T_0}^T w(\tau) \sup_{s \in K} |U(s+i\tau) - U_n(s+i\tau)| d\tau = 0,$$

$$\lim_{n \rightarrow \infty} \limsup_{T \rightarrow \infty} \frac{1}{U} \int_{T_0}^T w(\tau) \sup_{s \in K} |\xi_4(s+i\tau, \omega) - U_n(s+i\tau, \omega)| d\tau = 0$$

which are valid for any compact subset K of D_1 , we obtain that the measures $V_T^{(4)}$ and

$$\frac{1}{U} \int_{T_0}^T w(\tau) I_{\{\tau: \xi_4(s+i\tau, \omega) \in A\}} d\tau, \quad A \in \mathcal{B}(H(D_1)),$$

converge weakly to some measure V as $T \rightarrow \infty$ simultaneously. The last step of the proof consists of the checking that V coincides with P_{1, ξ_4} . The arguments used to show this are similar to those of [9], and involve elements of the ergodic theory.

Let S and S_1 be two metric spaces, and let $h : S \rightarrow S_1$ be a measurable function. Then every probability measure P on $(S, \mathcal{B}(S))$ induces a unique probability measure Ph^{-1} on $(S_1, \mathcal{B}(S_1))$ defined by $Ph^{-1}(A) = P(h^{-1}A)$, $A \in \mathcal{B}(S_1)$.

Lemma 5. *Let $h : S \rightarrow S_1$ be a continuous function, and let P_n and P be probability measures on $(S, \mathcal{B}(S))$. Suppose that P_n converges weakly to P as $n \rightarrow \infty$. Then $P_n h^{-1}$ converges weakly to Ph^{-1} as $n \rightarrow \infty$.*

PROOF. This is a special case of Theorem 5.1 of [5].

Now let

$$V_T^{(1)}(A) = \frac{1}{U} \int_{T_0}^T w(\tau) I_{\{\tau: \zeta^2(s+i\tau) \in A\}} d\tau, \quad A \in \mathcal{B}(H(D_1)),$$

$$V_T^{(2)}(A) = \frac{1}{U} \int_{T_0}^T w(\tau) I_{\{\tau: \zeta^2(2(s+i\tau)) \in A\}} d\tau, \quad A \in \mathcal{B}(H(D_1)),$$

and

$$\begin{aligned} \xi_1(s, \omega) &= \prod_p \left(1 - \frac{\omega(p)}{p^s}\right)^{-2}, \quad \omega \in \Omega, \quad s \in D_1, \\ \xi_2(s, \omega) &= \prod_p \left(1 - \frac{\omega^2(p)}{p^{2s}}\right)^{-2}, \quad \omega \in \Omega, \quad s \in D_1. \end{aligned}$$

Lemma 6. *The measures $V_T^{(1)}$ and $V_T^{(2)}$ converge weakly to the measures P_{1, ξ_1} and P_{1, ξ_2} , respectively, as $T \rightarrow \infty$.*

PROOF. The assertion of the lemma immediately follows from Lemmas 1, 2 and 5.

3. A fourdimensional limit theorem

In the previous section we have seen that the functions $\zeta^2(s)$, $\zeta^2(2s)$, $\zeta(2s-1)$ and $U(s)$ have a limit distribution in the space $H(D_1)$. In this section we will prove a joint limit theorem for these functions. We will use the following notation. We put

$$\begin{aligned} \xi_1(s) &= \zeta^2(s), & \xi_2(s) &= \zeta^2(2s), \\ \xi_3(s) &= \zeta(2s-1), & \xi_4(s) &= U(s), \end{aligned}$$

and denote by $v_j(m)$ the coefficients of the Dirichlet series of $\xi_j(s)$, $j = 1, \dots, 4$. Moreover, let

$$\begin{aligned} \Phi(s) &= (\xi_1(s), \dots, \xi_4(s)), \\ \Phi(s, \omega) &= (\xi_1(s, \omega), \dots, \xi_4(s, \omega)), \quad \omega \in \Omega, s \in D_1, \end{aligned}$$

where

$$\xi_j(s, \omega) = \sum_{m=1}^{\infty} v_j(m) \frac{\omega(m)}{m^s}.$$

Thus $\Phi(s, \omega)$ is an $H^4(D_1)$ -valued random element, where $H^4(D_1)$ denotes the Cartesian product of $H(D_1) \times H(D_1) \times H(D_1) \times H(D_1)$. Let as above P_Φ stand for the distribution of $\Phi(s, \omega)$, and define the probability measure

$$P_{T,w}^{(4)}(A) = \frac{1}{U} \int_{T_0}^T w(\tau) I_{\{\tau: \Phi(s+i\tau) \in A\}} d\tau, \quad A \in \mathcal{B}(H^4(D_1)).$$

Proposition. *The measure $P_{T,w}^{(4)}$ converges weakly to P_Φ as $T \rightarrow \infty$.*

We divide the proof of the proposition into three parts, and we state two first parts as individual lemmas. Before that we recall that the family of probability measures $\{P\}$ is relatively compact if every sequence of elements of $\{P\}$ contains a weakly convergent subsequence, and the family $\{P\}$ is tight if for an arbitrary $\varepsilon > 0$ there exists a compact set K such that $P(K) > 1 - \varepsilon$ for all P from $\{P\}$.

Lemma 7. *The family of probability measures $P_{T,w}^{(4)}$ is relatively compact.*

PROOF. By Lemmas 3, 4 and 6 we have that the probability measures $V_T^{(l)}$ converge weakly to the measures P_{1,ξ_l} , respectively, as $T \rightarrow \infty$, $l = 1, \dots, 4$. Consequently, the family of probability measures $\{V_T^{(l)}\}$ is relatively compact, $l = 1, \dots, 4$. The space of analytic functions $H(D_1)$ is a complete separable space. Hence it follows by the Prokhorov theorem (see, for example, [5], Theorem 6.2) that the family $\{V_T^{(l)}\}$ is tight, $l = 1, \dots, 4$. Thus for an arbitrary $\varepsilon > 0$ there exists a compact set $K_l \in H(D_1)$ such that

$$(5) \quad V_T^{(l)}(H(D_1) \setminus K_l) < \frac{\varepsilon}{4}, \quad l = 1, \dots, 4.$$

Let η_T be a random variable on $(\tilde{\Omega}, \mathcal{F}, \mathbb{P})$ such that

$$\mathbb{P}(\eta_T \in A) = \frac{1}{U} \int_{T_0}^T w(t) I_A dt, \quad A \in \mathcal{B}(\mathbb{R}),$$

and let

$$\begin{aligned}\xi_{l,T}(s) &= \xi_l(s + i\eta_T), \quad l = 1, \dots, 4, \\ \Phi_T(s) &= (\xi_{1,T}(s), \dots, \xi_{4,T}(s)).\end{aligned}$$

Then in view of (5) the definition of $V_T^{(l)}$ yields

$$(6) \quad \mathbb{P}(\xi_{l,T}(s) \in H(D_1) \setminus K_l) < \frac{\varepsilon}{4}, \quad l = 1, \dots, 4.$$

Now let us take $K = K_1 \times \dots \times K_4$. Then K is a compact set of the space $H^4(D_1)$, and in virtue of (6)

$$\begin{aligned}P_{T,w}^{(4)}(H^4(D_1) \setminus K) &= \mathbb{P}(\Phi_T(s) \in H^4(D_1) \setminus K) \\ &= \mathbb{P}\left(\bigcup_{l=1}^4 (\xi_{l,T}(s) \in H(D_1) \setminus K_l)\right) \\ &\leq \sum_{l=1}^4 \mathbb{P}(\xi_{l,T}(s) \in H(D_1) \setminus K_l) < \varepsilon.\end{aligned}$$

This shows that the family of probability measures $\{P_{T,w}^{(4)}\}$ is tight. Hence by the Prokhorov theorem (Theorem 6.1 of [5]) it is relatively compact.

Now let $s_1, \dots, s_n$ be arbitrary points on D_1 , and we set

$$\sigma_1 = \min_{1 \leq m \leq n} \operatorname{Re} s_m, \quad \sigma_2 = \max_{1 \leq m \leq n} \operatorname{Re} s_m.$$

Then we have $\sigma_1 > 3/4$. Moreover, let $\sigma_3 = 3/4 - \sigma_1 < 0$, $\sigma_4 = 1 - \sigma_2 > 0$ and $D_3 = \{s \in \mathbb{C} : \sigma_3 < \sigma < \sigma_4\}$. We take arbitrary complex numbers u_{lm} , and let the function $h : H^4(D_1) \rightarrow H(D_3)$ be given by the formula

$$h(f_1, \dots, f_4) = \sum_{l=1}^4 \sum_{m=1}^n u_{lm} f_l(s_m + s),$$

where $s \in D_3$, $f_l \in H(D_1)$, $l = 1, \dots, 4$. We set

$$W(s) = h(\xi_1(s), \dots, \xi_4(s)),$$

and denote the convergence in distribution by $\xrightarrow{\mathcal{D}}$.

Lemma 8. *The relation*

$$W(s + i\eta_T) \xrightarrow[T \rightarrow \infty]{\mathcal{D}} h(\Phi)$$

holds.

For the proof of Lemma 8 we need the following result.

Lemma 9. *Let for $\sigma > \sigma_0 + 1/2$ the function $f(s)$ be given by an absolutely convergent Dirichlet series*

$$\sum_{m=1}^{\infty} \frac{a_m}{m^s},$$

such that $\sum_{m \leq n} |a_m|^2 = Bn^{2\sigma_0}$. Suppose that $f(s)$ is a meromorphic function in the half-plane $\sigma > \sigma_0$, all poles in this region are included in a compact set, and for $\sigma \geq \sigma_0$

$$f(\sigma + it) = B|t|^\delta$$

with some positive δ . Moreover, suppose that the functions $w(\tau)$ and $f(s)$ satisfy the estimate

$$\int_{T_0}^T w(\tau) |f(\sigma + it + i\tau)| d\tau = BU(1 + |t|)^\beta$$

with some positive β for all $\sigma > \sigma_0$ and all $t \in \mathbb{R}$. Then the probability measure

$$\frac{1}{U} \int_{T_0}^T w(\tau) I_{\{\tau: f(s+i\tau) \in A\}} d\tau, \quad A \in \mathcal{B}(M(D_0)),$$

where $D_0 = \{s \in \mathbb{C} : \sigma > \sigma_0\}$, converges weakly to the distribution of the random element

$$\sum_{m=1}^{\infty} \frac{a_m \omega(m)}{m^s}, \quad \omega \in \Omega, \quad s \in D_0,$$

as $T \rightarrow \infty$.

PROOF. The lemma is a Theorem of [7] proved for the Matsumoto zeta-function which satisfies all conditions of the lemma. The general case of the lemma is identic to that of [7].

PROOF of Lemma 8. For $\sigma > \sigma_3 + 1/4$ we have

$$W(s) = \sum_{l=1}^4 \sum_{m=1}^n u_{lm} \xi_l(s_m + s).$$

Suppose that in this region

$$\xi_l(s) = \sum_{m=1}^{\infty} \frac{v_l(m)}{m^s}.$$

Thus

$$W(s) = \sum_{k=1}^{\infty} \frac{w_k}{k^s},$$

where

$$w_k = \sum_{l=1}^4 \sum_{m=1}^n \frac{u_{lm} v_l(k)}{k^{s_m}}.$$

By Lemma 4 of [10]

$$\int_{T_0}^T w(\tau) |\zeta(\sigma + it + i\tau)|^2 d\tau = BU(1 + |t|)^2$$

for $\sigma > 1/2$ and for all $t \in \mathbb{R}$. Consequently, the well-known properties of the Riemann zeta-function as well as of $U(s)$ and Lemma 9 with $\sigma_0 = 3/4$ yield the weak convergence of the probability measure

$$(7) \quad \frac{1}{U} \int_{T_0}^T w(\tau) I_{\{\tau: W(s+i\tau) \in A\}} d\tau, \quad A \in \mathcal{B}(H(D_3)),$$

to the measure P_W as $T \rightarrow \infty$, where P_W is the distribution of the random element

$$W(s, \omega) = \sum_{k=1}^{\infty} \frac{w_k \omega(k)}{k^s}, \quad \omega \in \Omega, \quad s \in D_3.$$

On the other hand,

$$\begin{aligned} W(s, \omega) &= \sum_{l=1}^4 \sum_{m=1}^n u_{lm} \sum_{k=1}^{\infty} \frac{w_k \omega(k)}{k^{s_m + s}} = \\ &= \sum_{l=1}^4 \sum_{m=1}^n u_{lm} \xi_l(s_m + s, \omega) = h(\Phi(s, \omega)). \end{aligned}$$

Thus the measure (7) converges weakly to P_W as $T \rightarrow \infty$. This proves the lemma.

PROOF of the Proposition. By Lemma 7 there exists a sequence $T_1 \rightarrow \infty$ such that the measure $P_{T_1, w}^{(4)}$ converges weakly to some probability measure P as $T_1 \rightarrow \infty$. Suppose that P is the distribution of an $H^4(D_1)$ -valued random element

$$\Phi_1(s) = (\xi_{11}(s), \dots, \xi_{14}(s)).$$

Then, clearly,

$$(8) \quad \Phi_{T_1} \xrightarrow[T_1 \rightarrow \infty]{\mathcal{D}} \Phi_1.$$

Taking into account the continuity of the function h , hence and from Lemma 5 we deduce that

$$h(\Phi_{T_1}) \xrightarrow[T_1 \rightarrow \infty]{\mathcal{D}} h(\Phi_1).$$

Therefore, by the definition of W

$$(9) \quad W(s + i\eta_T) \xrightarrow[T_1 \rightarrow \infty]{\mathcal{D}} h(\Phi_1).$$

By Lemma 8

$$W(s + i\eta_T) \xrightarrow[T_1 \rightarrow \infty]{\mathcal{D}} h(\Phi).$$

Hence, and from (9)

$$(10) \quad h(\Phi) \stackrel{\mathcal{D}}{=} h(\Phi).$$

Now let $h_1 : H(D_3) \rightarrow \mathbb{C}$ be defined by the formula

$$h_1(f) = f(0), \quad f \in H(D_3).$$

This function, clearly, is measurable. Therefore (10) implies the relation

$$h(\Phi)(0) \stackrel{\mathcal{D}}{=} h(\Phi_1)(0).$$

Thus by the definition of h we find that

$$(11) \quad \sum_{l=1}^4 \sum_{m=1}^n u_{lm} \xi_l(s_m) \stackrel{\mathcal{D}}{=} \sum_{l=1}^4 \sum_{m=1}^n u_{lm} \xi_{1l}(s_m)$$

for arbitrary complex numbers u_{lm} . The hyperplanes in the space $\mathbb{R}^{8n}$ form a determining class (see [5]). Therefore, the hyperplanes also form a determining class in the space $\mathbb{C}^{4n}$. Thus, taking into account (11), we obtain that $\mathbb{C}^{4n}$ -valued random elements $\xi_l(s_m)$ and $\xi_{1l}(s_m)$, $l = 1, \dots, 4$, $m = 1, \dots, n$, have the same distribution.

Now let K be a compact subset of the strip D_1 , and let $f_1, \dots, f_4 \in H(D_1)$. For an arbitrary $\varepsilon > 0$ we set

$$G = \{(g_1, \dots, g_4) \in H^4(D_1) : \sup_{s \in K} |g_l(s) - f_l(s)| \leq \varepsilon, l = 1, \dots, 4\},$$

and we choose a sequence $\{s_m\}$ to be dense in K . Moreover, let

$$G_n = \{(g_1, \dots, g_4) \in H^4(D_1) : |g_l(s_m) - f_l(s_m)| \leq \varepsilon, \\ l = 1, \dots, 4, m = 1, \dots, n\}.$$

Thus the above mentioned properties of the random elements $\xi_l(s_m)$ and $\xi_{1l}(s_m)$ show that

$$(12) \quad m_H(\omega \in \Omega : \Phi(s, \omega) \in G_n) = P(\Phi_1(s) \in G_n).$$

Since the sequence $\{s_m\}$ is dense in K , we have $G_n \rightarrow G$ as $n \rightarrow \infty$. Thus, letting $n \rightarrow \infty$ in (12), we find

$$(13) \quad m_H(\omega \in \Omega : \Phi(s, \omega) \in G) = P(\Phi_1(s) \in G).$$

The space $H^4(D_1)$ is separable. Therefore, finite intersections of the spheres form a determining class (see [5]). Hence we obtain from (12) and (13) that

$$\Phi \stackrel{\mathcal{D}}{=} \Phi_1.$$

This and (8) yield

$$(14) \quad \Phi_{T_1} \xrightarrow[T_1 \rightarrow \infty]{\mathcal{D}} \Phi.$$

This means that the measure $P_{T,w}^{(4)}$ converges weakly to the distribution of the random element Φ as $T_1 \rightarrow \infty$. Now the assertion of the Proposition follows from Lemma 7, since the random element Φ in (14) is independent of the choice of the sequence T_1 .

4. Proof of the Theorem

Let the function $h : H^4(D_1) \rightarrow H(D_1)$ be given by the formula

$$h(f_1, f_2, f_3, f_4) = f_1 f_2 f_3 f_4, \quad f_1, \dots, f_4 \in H(D_1).$$

Since the latter function is continuous, the assertion of the Theorem follows from the Proposition and Lemma 5.

The Corollary is the Theorem with $w(\tau) \equiv 1$.

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