Publ. Math. Debrecen
52 / 3-4 (1998), 611–633

On the strong summability of Walsh series

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Dedicated to the 60th birthday
of Professors Zoltán Daróczy and Imre Kátai

Abstract. In this paper we investigate the strong (H, p)- and BMO-summability of Walsh-Fourier series. Among others we give a characterization of points in which the Walsh-Fourier series of an integrable function is (H, p)- and BMO-summable. This is the analogue of Gabisonia's result that characterizes the points of strong summability with respect to the trigonometric system.

1. Introduction

It was proved by L. Fejér [3] that the $(C, 1)$ means of the trigonometric Fourier series (TFS) of any 2π periodic continuous function converges uniformly to the function. The same problem for integrable functions was investigated by H. Lebesgue [7]. He proved that the TFS of any integrable function $f \in L^1_{2\pi}$ is a.e. $(C, 1)$-summable, i.e.

$$\frac{1}{n}\sum_{k=0}^{n-1}\big[(S_k^T f)(x) - f(x)\big] \to 0 \quad \text{as} \quad n \to \infty \tag{1.1}$$

$$(\text{for a.e. } x \in (-\pi, \pi)).$$

Mathematics Subject Classification: 42C10, 43A55, 40F05.
Key words and phrases: Walsh series, strong summability, Lebesgue points, BMO-means.
This research was supported by the grants OTKA T 020497 and FKFP 0204/97.

Lebesgue gave the following simple sufficient condition for the points in which (1.1) holds. Namely, he showed that the limit relation holds in every point $x \in (-\pi, \pi)$ for which

$$(\Lambda_n f)(x) := \frac{1}{|J_n(x)|} \int_{J_n(x)} |f(x) - f(s)|\, ds \to 0 \quad (n \to \infty), \tag{1.2}$$

where $J_n(x) := [x - \pi 2^{-n}, x + \pi 2^{-n})$ and $|J_n(x)|$ is the lenght of $J_n(x)$. Such points are called *Lebesgue points of the function* f For any $f \in L^1_{2\pi}$ almost every x is a Lebesgue point of f.

Strong summability, i.e. the convergence of the strong means

$$\left(\frac{1}{n} \sum_{k=0}^{n-1} |(S_k^T f)(x) - f(x)|^p \right)^{1/p} \qquad (x \in \mathbb{R},\ n \in \mathbb{N}^*,\ p > 0)$$

was first considered by G. H. Hardy and J. E. Littlewood [6]. They showed that for any $f \in L^r_{2\pi}$ $(1 < r < \infty)$ the strong means tend to 0 a.e. if $n \to \infty$.

Let us consider it more generally. We will introduce strong means generated by the stricly increasing continuous function $\Psi : [0, +\infty) \to [0, +\infty)$ with $\Psi(0) = 0$. Then the *Hardy operators* are defined as

$$(H_n^{T,\Psi} f)(x) := \Psi^{-1} \left(2^{-n} \sum_{k=0}^{2^n - 1} \Psi\left(|(S_k^T f)(x)| \right) \right) \quad (x \in \mathbb{R},\ n \in \mathbb{N}), \tag{1.3}$$

where Ψ^{-1} is the inverse of the function Ψ. If $\Psi(t) = t^p$ $(0 \le t < \infty$, $0 < p < \infty)$ then we use the simpler notation $H_n^{T,p}$. The trigonometric Fourier series of $f \in L^1_{2\pi}$ is called (H, Ψ)-summable at $x \in \mathbb{R}$ if

$$\lim_{n\to\infty} \left(H_n^{T,\Psi}(f - f(x)) \right)(x) = 0. \tag{1.4}$$

If $\Psi(t) = t^p$ $(0 < p < \infty,\ t \ge 0)$ then the shorter notation (H, p)-summability will be used. The (H, p)-means increases with p, therefore (H, p) $(p \ge 1)$-summability implies $(H, 1)$-summability and hence the convergence of the $(C, 1)$-means follows.

For functions in $L^1_{2\pi}$ the (H, p)-summability was investigated by J. Marcinkiewicz [8] for $p = 2$, and later by A. Zygmund [15] for the general case. He proved that (1.4) holds a.e. for $\Psi(t) = t^p$ $(0 < p < \infty$, $t \ge 0)$, as $n \to \infty$.

For the points in which the strong means tend to 0 O. D. Gabisonia gave a simple sufficient condition (see [5], [10], [11]). Namely, modifying the definition of $\Lambda_n f$ he introduced the following operator

$$(1.5)\qquad (\Lambda_n^{(p)} f)(x) := \left(\sum_{t \in T_n} \left(\frac{1}{t} \int_{J_n(x+t)} |f(s) - f(x)| ds \right)^p \right)^{1/p},$$

where $p > 0$ and $T_n := \{(k + 1/2)2\pi 2^{-n} : -2^{n-1} \le k < 2^{n-1},\ k \in \mathbb{Z}\}$.

Gabisonia [5] showed that the Hardy-operators can be estimated by the $\Lambda_n^{(p)}$'s, i.e.

$$(1.6)\qquad \left(H_n^{T,p}(f - f(x))\right)(x) \le C_p \left(\Lambda_n^{(p)} f\right)(x) \quad (p > 1).$$

Moreover (see [5], [10], [11]),

$$(1.7)\qquad \left(\Lambda_n^{(p)} f\right)(x) \to 0, \quad \text{if} \quad n \to \infty$$

for a.e. $x \in \mathbb{R}$. The points x satisfying (1.7) are called *Gabisonia-points* or *strong Lebesgue-points* of the function f. A.e. x point is a strong Lebesgue-point for f therefore the result of Zygmund, the trigonometric Fourier-series of any integrable function is a.e. (H, p) summable $(0 < p < \infty)$, follows by (1.6). Since $\Lambda_n f = O(1)\Lambda_n^{(p)} f$ $(p \ge 1, n > 0)$ we have that every Gabisonia-point is a Lebesgue-point for f and this justifies the notion. V. A. Rodin [10], [11] generalized these results for certain Ψ-means, and BMO-means. Moreover, his idea to consider BMO-means was an essential contribution to this subject.

In this paper we investigate the similar question for Walsh-Fourier series. In Section 2 we introduce the dyadic analogue of Lebesgue- and strong Lebesgue-points and summarize the results. It turns out that for shift-invariant systems the (H, p) summability methods are a.e. equivalent to each others for any $p > 0$. Thus it is enough to investigate the $(H, 2)$ summability (see Section 3).

In Section 4 we estimate the maximal operator of the strong $(H, 2)$-means of Walsh-Fourier series by the maximal operator of dyadic Gabisonia operators. In Section 5 we show that this operator is of weak type (1,1) (in a little sharper sense as usual). This can be used to derive an L^1-norm estimation for this maximal function.

2. Strong means of Walsh-Fourier series

The analogue of Lebesgue's theorem for Walsh-Fourier series was proved by N. J. Fine [4]. We note that in this case the Lebesgue characterizations cannot be used. Namely, it follows from a result of D. K. Faddeeff [2] (see also Alexits [1]) that there exists an integrable function with a Lebesgue point such that the Walsh-Fourier series (WFS) of this function is not $(C,1)$ summable at this point. The analogue notion of the Lebesgue point for the Walsh-system is the following. Denote $I_n(t)$ the dyadic interval of lenght 2^{-n} containing $t \in \mathbb{I} := [0,1)$ and set $e_k := 2^{-k-1}$ $(k \in \mathbb{N})$. The point $x \in \mathbb{I}$ is called *Walsh-Lebesgue point (WLP) of* $f \in L^1 := L^1[0,1)$ if

$$(W_n f)(x) := \sum_{k=0}^{n} 2^k \int_{I_n(x \dot{+} e_k)} |f(s) - f(x)|\, ds \to 0, \quad \text{as} \quad n \to \infty, \tag{2.1}$$

where $\dot{+}$ denotes the dyadic addition (see [13]).

It is known (see [13]) that if $f \in L^1$ then almost every point is Walsh-Lebesgue point for f. Furthermore, the WFS are $(C,1)$-summable in the Walsh-Lebesgue points.

The convergence of sequences of singular integral operators in Walsh-Lebesgue points was investigated by F. Weisz [14].

The analogues of the results of Marcinkiewicz and Zygmund for the Walsh-system was proved by F. Schipp [12] for $p = 2$. The general case and the case of BMO-means was proved by V. A. Rodin [11]. In this paper – similarly to Gabisonia's result – we give a sufficient condition for the (H,p)-summability of WFS. This condition can be obtained from (2.1) in a similar way as we get the Gabisonia condition from the definition of Lebesgue points.

On the basis of (1.5) and (2.1) we introduce the operators

$$(W_n^{(p)} f)(x) := \left(\sum_{t \in Q_n} \left(\sum_{k=0}^{n-1} 2^k \chi_{[0,2^{-k})}(t) \int_{I_n(x \dot{+} t \dot{+} e_k)} |f(x) - f(s)| ds \right)^p \right)^{1/p}$$

$$(n \in \mathbb{N},\ x \in \mathbb{I},\ p > 0), \tag{2.2}$$

where $Q_n := \{k2^{-n} : k = 0, 1, 2, \ldots, 2^n - 1\}$ and χ_H denotes the characteristic function of H. For $n \in \mathbb{N}$ let us introduce the projections

$$(2.3) \qquad (E_n f)(x) := (S_{2^n}^W f)(x) = 2^n \int_{I_n(x)} f(s)\, ds \quad (f \in L^1,\ x \in \mathbb{I})$$

and the operators

$$\begin{aligned}(V_n^{(p)} g)(x) :&= \left(\sum_{t \in Q_n} \left| \sum_{k=0}^{n} 2^{k-n} \chi_{[0,2^{-k})}(t)(E_n g)(x \dotplus t \dotplus e_k) \right|^p \right)^{1/p} \\ (2.4) \qquad &= 2^{-n/q} \left\| \sum_{k=0}^{n} 2^k \chi_{[0,2^{-k})} \tau_{e_k \dotplus x} E_n g \right\|_p \\ &\qquad (g \in L^1,\ x \in \mathbb{I},\ p > 0),\end{aligned}$$

where $(\tau_s h)(x) := h(x \dotplus s)$ is the dyadic translation operator and $1/p + 1/q = 1$.

We shall say that the point $x \in \mathbb{I}$ is a *strong Walsh–Lebesgue point* (SWLP) for $f \in L^1$ if

$$(2.5) \qquad \lim_{n\to\infty} (W_n^{(p)} f)(x) = \lim_{n\to\infty} \left(V_n^{(p)}(|f - f(x)|) \right)(x) = 0.$$

By (2.1) and (2.2) we have $W_n f \le W_n^{(p)} f$ $(n \in \mathbb{N},\ p \ge 1)$. Consequently, every SWLP is a WLP.

The Hardy-operator with respect to the Walsh system will be denoted by $H_n^{W,p}$. We will show that for any function $f \in L^1$ the $H_n^{W,2} f$ means can be estimated by $V_n^{(2)} f$. Set

$$(2.6) \qquad H^{W,p} f := \sup_n H_n^{W,p} f, \quad V^{(p)} f := \sup_n V_n^{(p)} f.$$

We shall prove the following inequality for these maximal operators.

Theorem 1. *The maximal operator of the Hardy-operators $H_n^{W,2}$ satisfies*

$$(2.7) \qquad H^{W,2} f \le 2V^{(2)}(|f|) \quad (f \in L^1).$$

In Section 5 we show that the operator $V^{(2)}$ is of type (∞, ∞) and of weak type (1,1) in the following sharp form.

Theorem 2. *i) For any function* $f \in L^\infty$

$$\|V^{(2)}f\|_\infty \le 2\|f\|_\infty. \tag{2.8}$$

ii) For any $f \in L^1$ *and* $y > 0$ *we have*

$$\left|\left\{x \in \mathbb{I} : (V^{(2)}f)(x) > 5y\right\}\right| \le \frac{321}{y}\int_{\{E^*|f|>y\}} |f(s)|\,ds \le \frac{321}{y}\|f\|_1, \tag{2.9}$$

where $E^*f = \sup_n |E_n f|$ *is the dyadic maximal operator.*

Hence by Marcinkiewicz's interpolation theorem we get

Corollary 1. *For any function* $f \in L^p$ $(1 < p \le \infty)$

$$\|V^{(2)}f\|_p \le C_p\|f\|_p, \tag{2.10}$$

where C_p *depends only on* p.

We remark that (2.10) can be obtained immediately from (2.9) without applying Marcinkiewicz's interpolation theorem. The same argument yields the following estimation for the L^1-norm of $V^{(2)}f$

Corollary 2. *For the integral of* $V^{(2)}f$ *we have*

$$\|V^{(2)}f\|_1 \le C\left(\|f\|_1 + \int_0^1 |f(s)|\log\frac{(E^*|f|)(s)}{\|f\|_1}\,ds\right). \tag{2.11}$$

3. Estimation for the BMO-means

After having introduced (H,p) and (H,Ψ)-means now we introduce the BMO-means. To this end set

$$\mathcal{J} := \{J := [k2^n, (k+1)2^n) \cap \mathbb{N} : k, n \in \mathbb{N}\}. \tag{3.1}$$

Then $\mathcal{J}$ is the collection of integer dyadic intervals. The number of elements in $J \in \mathcal{J}$ will be denoted by $|J|$. The mean value of the sequence $s = (s_k, k \in \mathbb{N})$ with respect to J is denoted by

$$s^J := \frac{1}{|J|}\sum_{k\in J} s_k.$$

The BMO norm of the sequence s is defined by

$$\|s\|_{BMO} := \sup_{J\in\mathcal{J}} \Omega_J := \sup_{J\in\mathcal{J}} \left(|J|^{-1} \sum_{k\in J} \left|s_k - s^J\right|^2 \right)^{1/2}. \tag{3.2}$$

This norm is in strong connection with the BMO-norm of functions. Namely, denote $s^\diamondsuit$ the step function on $[0,\infty)$ having the value s_n on the interval $[n, n+1)$. Fix the number $N \in \mathbb{N}$ and set

$$(s_N^\diamondsuit)(t) := s^\diamondsuit(2^N t) \quad (t \in \mathbb{I}).$$

It is easy to see that

$$\|s\|_{BMO} = \sup_N \|s_N^\diamondsuit\|_{BMO}, \tag{3.3}$$

where on the right hand side we take the usual dyadic BMO-norm of the function $s_N^\diamondsuit$. This connection can be used to deduce the properties of this sequence norm. For example, if L^Ψ denotes the Orlicz-space generated by the function $\Psi(t) := \exp(|t|) - 1$ $(t \in \mathbb{R})$ then $BMO \subset L^\Psi$ and

$$\|f\|_{L^\Psi} \le C\|f\|_{BMO} \quad (f \in BMO), \tag{3.4}$$

where $C > 0$ is an absolute constant. Furthermore it is known, that L^Ψ is the minimal rearrangement invariant subspace in L^1 containing BMO.

The 2^N-th (H,p) mean of s corresponds to the L^p-norm of the function $s_N^\diamondsuit$:

$$\left(2^{-N} \sum_{k=0}^{2^N-1} |s_k|^p \right)^{1/p} = \|s_N^\diamondsuit\|_p \quad (p > 0).$$

It is known, that

$$\|f\|_p \le C_p \|f\|_{BMO} \quad (f \in BMO,\ 1 \le p < \infty),$$

where the constant $C_p = O(p)$ does not depend on f. This implies

$$s^{\langle p\rangle} := \sup_N \left(2^{-N} \sum_{k=0}^{2^N-1} |s_k|^p \right)^{1/p} \le C_p \|s\|_{BMO} \quad (1 \le p < \infty). \tag{3.5}$$

From (3.4) and (3.5) it follows that all of the mentioned means can be estimated from above by the BMO-means. In the case of Fourier series with respect to certain orthogonal systems a lower estimation is also true. Suppose that the system $\epsilon = (\epsilon_n, n \in \mathbb{N})$ is orthonormal with respect to the scalar product $\langle \cdot, \cdot \rangle$, $|\epsilon_n| \le 1$ and has the following *shift-property*: For every $J = [k, k+2^s) \cap \mathbb{N} \in \mathcal{J}$

$$\epsilon_{k+\ell} = \epsilon_k \epsilon_\ell \quad (0 \le \ell < 2^s). \tag{3.6}$$

For example the complex trigonometric system and the Walsh-system satisfy (3.6). The k-th partial sum of the Fourier series with respect to the system ϵ will be denoted by

$$S_k^\epsilon f := \sum_{\ell=0}^{k-1} \langle f, \epsilon_\ell \rangle \epsilon_\ell \quad (k \in \mathbb{N}^*), \tag{3.7}$$

where by definition $S_0^\epsilon f = 0$.

First we show that (3.6) implies

$$S_{k+\ell}^\epsilon f - S_k^\epsilon f = \epsilon_k S_\ell^\epsilon (f \overline{\epsilon}_k) \quad (0 \le \ell < 2^s,\ [k, k+2^s) \in \mathcal{J}). \tag{3.8}$$

Indeed,

$$S_{k+\ell}^\epsilon f - S_k^\epsilon f = \sum_{j \in [k, k+\ell)} \langle f, \epsilon_j \rangle \epsilon_j = \epsilon_k \sum_{i \in [0,\ell)} \langle f \overline{\epsilon}_k, \epsilon_i \rangle \epsilon_i = \epsilon_k S_\ell^\epsilon (f \overline{\epsilon}_k).$$

Hence for the means

$$\Omega_J^\epsilon f := \left(|J|^{-1} \sum_{j \in J} \left| S_j^\epsilon f - 2^{-s} \sum_{i \in J} S_i^\epsilon f \right|^2 \right)^{1/2} \qquad (J = [k, k+2^s) \in \mathcal{J}) \tag{3.9}$$

it follows that

$$\Omega_{[k,k+2^s)}^\epsilon f = \Omega_{[0,2^s)}^\epsilon (f \overline{\epsilon}_k) \quad ([k, k+2^s) \in \mathcal{J}). \tag{3.10}$$

In order to see this apply (3.8) for $\ell < 2^s$. Then

$$\begin{aligned} S_{k+\ell}^\epsilon f - 2^{-s} \sum_{j \in J} S_j^\epsilon f &= (S_{k+\ell}^\epsilon f - S_k^\epsilon f) - 2^{-s} \sum_{j \in J} (S_j^\epsilon - S_k^\epsilon f) \\ &= \epsilon_k \left(S_\ell^\epsilon (f \overline{\epsilon}_k) - 2^{-s} \sum_{i=0}^{2^s - 1} S_i^\epsilon (f \overline{\epsilon}_k) \right). \end{aligned}$$

Hence

$$\Omega^{\epsilon}_{[k,k+2^s)}f = \left(2^{-s}\sum_{\ell=0}^{2^s-1}\left|S^{\epsilon}_{\ell}(f\overline{\epsilon}_k) - 2^{-s}\sum_{i=0}^{2^s-1}S^{\epsilon}_i(f\overline{\epsilon}_k)\right|^2\right)^{1/2} = \Omega^{\epsilon}_{[0,2^s)}(f\overline{\epsilon}_k).$$

The maximal operator of Ψ-means and BMO-means with respect to the system ϵ are denoted by

$$H^{\epsilon,\Psi}f := \sup_n H^{\epsilon,\Psi}_n f, \quad H^{\epsilon,BMO}f := \sup_{J\in\mathcal{J}}\Omega^{\epsilon}_J f.$$

From (3.4) and (3.5) it follows that

$$2^{-n}\sum_{k=0}^{2^n-1}|S^{\epsilon}_k f|^p \le C_p 2^{-n}\sum_{k=0}^{2^n-1}(\exp(|S^{\epsilon}_k f|)-1). \tag{3.11}$$

Obviously

$$\Omega^{\epsilon}_{[0,2^n)}f \le H^{\epsilon,2}_n f.$$

Consequently, if the system ϵ satisfies (3.6) then by (3.10) we get the following reverse inequality

$$H^{\epsilon,BMO}f \le \sup_k H^{\epsilon,2}(f\overline{\epsilon}_k). \tag{3.12}$$

In connection with this inequality we introduce the following notion. Suppose that the operators H and V map functions defined on $\mathbb{I}$ into functions. We say that the operator V is an *absolute majorant* of H if for every $f\in\mathcal{D}_H$ we have that $|f|\in\mathcal{D}_V$ and $|Hf|\le V|f|$. Obviously every positive linear operator is an absolute majorant for itself. From (1.6) and from (2.7) and (2.9) it follows that the maximal operator of the Hardy-operators both in the trigonometric case and in the Walsh case has *an absolute majorant with weak type* (1,1). Using this concept we obtain from (3.11) and (3.12) the next

Equvivalence Principle. *Suppose that the complete unitary orthonormal system ϵ satisfies* (3.6). *If the maximal operator $H^{\epsilon,2}$ has an absolute majorant of weak type $(1,1)$ then for any function $f\in L^1$ the Fourier series of f with respect to the system ϵ is a.e. $(H,2)$ summable. Moreover in this case the (H,p) $(1\le p<\infty)$, (H,Ψ) $(\Psi(t)=\exp(t)-1)$ and BMO summabilities are equivalent in the a.e. convergence sense.*

Especially, Theorem 1 and 2 implies for the Walsh-system

Corollary 3. *i) If $f \in L^1$ and $0 < p < \infty$, then*

$$\lim_{n\to\infty} \big(H_n^{W,p}(f - f(x))\big)(x) = 0 \quad \textit{for a.e.} \quad x \in \mathbb{I}.$$

ii) Let $f \in L^1$ and $\Psi_\lambda(t) := \exp(t/\lambda) - 1$ $(t \geq 0,\ \lambda > 0)$. Then there exists λ_0 such that for every number $\lambda > \lambda_0$

$$\lim_{n\to\infty} (H_n^{W,\Psi_\lambda}(f - f(x)))(x) = 0 \quad \textit{for a.e.} \quad x \in \mathbb{I}.$$

4. Pointwise estimation for strong means

In order to show (2.7) we need the Walsh-Dirichlet kernels that are denoted by

$$D_0 := 0,\ D_m := D_m^W := \sum_{k=0}^{m-1} w_k \quad (m \in \mathbb{N}^*).$$

First we prove the identity

$$D_m(t) = (d_n^- w_m)(t) \quad (t \in [2^{-n-1}, 2^{-n}),\ n, m \in \mathbb{N}), \tag{4.1}$$

where

$$(d_n^- g)(t) := \sum_{k=0}^{n-1} 2^{k-1} \left(g(t) - g(t \dotplus e_k)\right) - 2^{n-1} \left(g(t) - g(t \dotplus e_n)\right)$$
$$(t \in \mathbb{I},\ n \in \mathbb{N}^*) \tag{4.2}$$

is the n-th *modified dyadic difference operator*. Indeed, from the definition of the Walsh-functions it follows that

$$2^{k-1} \left(w_m(t) - w_m(t \dotplus e_k)\right) = 2^{k-1}(1 - (-1)^{m_k}) w_m(t) = 2^k m_k w_m(t)$$
$$(t \in \mathbb{I},\ k \in \mathbb{N}).$$

Hence

$$d_n^- w_m = w_m \left(\sum_{k=0}^{n-1} m_k 2^k - m_n 2^n \right).$$

It is known (see [13]), that D_m can be written in the following form

$$D_m = w_m \sum_{j=0}^{\infty} m_j w_{2^j} D_{2^j} \qquad \left(m = \sum_{j=0}^{\infty} m_j 2^j \in \mathbb{N} \right).$$

Since [13]

$$w_{2^j}(t) D_{2^j}(t) = \begin{cases} 2^j, & t \in [0, 2^{-j-1}), \\ -2^j, & t \in [2^{-j-1}, 2^{-j}), \\ 0, & t \in [2^{-j}, 1), \end{cases}$$

we have that

$$D_m(t) = w_m \left(\sum_{k=0}^{n-1} m_k 2^k - m_n 2^n \right) = (d_n^- w_m)(t)$$

$$\big(t \in [2^{-n-1}, 2^{-n}),\ n, m \in \mathbb{N}\big)$$

and (4.1) is proved.

Denote

$$(f \star g)(x) := \int_0^1 f(x \dotplus t) g(t)\, dt = \langle \tau_x f, g \rangle \quad (x \in \mathbb{I}) \tag{4.3}$$

the dyadic convolution of the functions $f \in L^1, g \in L^\infty$. Starting from the representation (2.4) of $V_n^{(2)}$ we prove inequality (2.7).

Proof of Theorem 1. Since $S_{2^n}^W f = E_n f$ we have

$$S_m^W f = S_m^W (E_n f) = (E_n f) \star D_m \quad (m \le 2^n).$$

Let the characteristic function of the interval $[2^{-j-1}, 2^{-j})$ be denoted by χ_j $(j \in \mathbb{N})$. Using (4.1) we can write the function D_m in the form

$$D_m = \sum_{k=0}^{n-1} \chi_k d_k^- w_m + m \chi_{[0, 2^{-n})} \quad (0 \le m < 2^n).$$

Introducing the notations

$$\tag{4.4}\begin{aligned}\Delta_k^- g &:= d_k^- g + \frac{1}{2} g = -\sum_{j=0}^{k-1} 2^{j-1}\tau_{e_j} g + 2^{k-1}\tau_{e_k} g,\\ L_n g &:= \sum_{k=0}^{n-1} \chi_k \Delta_k^- g,\end{aligned}$$

we obtain the following representation of the Dirichlet kernels:

$$\begin{aligned}D_m &= \sum_{k=0}^{n-1} \chi_k \Delta_k^- w_m - \frac{1}{2} w_m + (m+1/2)\chi_{[0,2^{-n})}\\ &= L_n w_m - \frac{1}{2} w_m + (m+1/2)\chi_{[0,2^{-n})}.\end{aligned}$$

Hence

$$S_m^W f = (E_n f) \star (L_n w_m) - \frac{1}{2} f \star w_m + (m+1/2) 2^{-n} E_n f.$$

Thus for the $(H,2)$ means we have

$$\tag{4.5} (H_n^{W,2} f)(x) \le \left(2^{-n} \sum_{m=0}^{2^n-1} |\langle \tau_x E_n f, L_n w_m\rangle|^2 \right)^{1/2} + \frac{3}{2}(E^*|f|)(x).$$

There is a suitable vector

$$(a_0(x), a_1(x), \ldots, a_{2^n-1}(x)) \in \mathbb{R}^{2^n}, \quad \sum_{k=0}^{2^n-1} |a_k(x)|^2 = 1$$

such that the first term, without the factor $2^{-n/2}$, can be written in the form

$$\sigma_1(x) := \left(\sum_{m=0}^{2^n-1} |\langle \tau_x E_n f, L_n w_m\rangle|^2 \right)^{1/2} = \sum_{m=0}^{2^n-1} a_m(x) \langle \tau_x E_n f, L_n w_m\rangle$$

$$= \left\langle \tau_x E_n f, L_n \left(\sum_{m=0}^{2^n-1} a_m(x) w_m \right) \right\rangle = \langle \tau_x E_n f, L_n P_x\rangle = \langle L_n^\star \tau_x E_n f, P_x\rangle.$$

Here $L_n^\star$ is the adjoint of L_n and the Walsh polinomial $P_x = \sum_{m=0}^{2^n-1} a_m(x) w_m$ satisfies $\|P_x\|_2 = 1$. Applying Cauchy's inequality we get

$$\sigma_1(x) \leq \|L_n^\star \tau_x E_n f\|_2. \tag{4.6}$$

The operators Δ_k^- are self-adjoint, therefore

$$L_n^\star g = \sum_{k=0}^{n-1} \Delta_k^-(\chi_k g).$$

Hence we have the following estimation for $L_n^\star g$:

$$\begin{aligned} |L_n^\star g| &\leq \sum_{k=0}^{n-1} |\Delta_k^-(\chi_k g)| \leq \sum_{k=0}^{n-1} \sum_{j=0}^{k} 2^{j-1} \tau_{e_j}(\chi_k |g|) \\ &= \sum_{j=0}^{n-1} 2^{j-1} \tau_{e_j} \left(\sum_{k=j}^{n-1} \chi_k |g| \right). \end{aligned}$$

Clearly,

$$\sum_{k=j}^{n-1} \chi_k \leq \chi_{[0,2^{-j})}, \quad \tau_{e_j} \chi_{[0,2^{-j})} = \chi_{[0,2^{-j})}.$$

Consequently,

$$|L_n^\star g| \leq \sum_{j=0}^{n-1} 2^{j-1} \chi_{[0,2^{-j})} \tau_{e_j} |g|. \tag{4.7}$$

It follows from (2.4) that $E_n|f| \leq V_n^{(2)}|f|$, therefore by (4.4), (4.5), (4.6) and (4.7) we have

$$(H_n^{W,2} f)(x) \leq \frac{1}{2}(V_n^{(2)}|f|)(x) + \frac{3}{2}(E^*|f|)(x) \leq 2(V^{(2)}|f|)(x).$$

Hence (2.7) follows by taking the supremum.

5. The maximal operator of the Walsh–Gabisonia operators

In this section we prove Theorem 2. To this end we shall use the *Calderon–Zygmund decomposition* in the following form (see [13]).

Calderon–Zygmund lemma. *Let $f \in L^1$ and $y > \|f\|_1$. Then there exist a sequence of pairwise disjoint intervals $J_k \subseteq \mathbb{I}$ $(k \in \mathbb{N}^*)$ and a decomposition $f = \sum_{k=0}^{\infty} f_k$ of the function f such that :*

$$
\begin{aligned}
&i) && \|f_0\|_\infty \leq 2y,\\
&ii) && \{f_k \neq 0\} \subseteq J_k,\\
&iii) && \int_{J_k} f_k(s)ds = 0,\\
&iv) && |J_k|^{-1}\int_{J_k} |f_k(s)|ds \leq 4y \quad (k \in \mathbb{N}^*),\\
&v) && \sum_{j=1}^{\infty} |J_j| \leq \frac{1}{y}\int_U |f(s)|ds,\\
&vi) && U := \bigcup_{j=1}^{\infty} J_j = \{x \in \mathbb{I} : (E^*|f|)(x) > y\}.
\end{aligned}
\tag{5.1}
$$

We shall estimate the maximal operator $V^{(2)}f$ on the complementer of the set U by *generalized convolution operators*. In connection with this we prove

Lemma 1. *Let $\mathcal{I} = (J_k,\ k \in \mathbb{N}^*)$ be a system of pairwise disjoint dyadic intervals and let $\varphi_k \in L^1$ $(k \in \mathbb{N}^*)$ be a sequence of functions satisfying*

$$M := \sup_k \|\varphi_k\|_1 < \infty.$$

Then the generalized convolution operator

$$Tf := \sum_{k=1}^{\infty} (\chi_{J_k} f) \star \varphi_k \tag{5.2}$$

satisfies

$$\|Tf\|_1 \leq M\|\chi_U f\|_1 \quad (f \in L^1), \tag{5.3}$$

where $U := \bigcup_{k=1}^{\infty} J_k$.

Proof. Using the inequality

$$\|g \star h\|_1 \leq \|g\|_1\|h\|_1 \quad (g, h \in L^1)$$

we get that the series (5.2) converges in L^1–norm and

$$\|Tf\|_1 \leq \sum_{k=1}^{\infty} \|\chi_{J_k} f\|_1 \|\varphi_k\|_1 \leq M \sum_{k=1}^{\infty} \|\chi_{J_k} f\|_1 = M\|\chi_U f\|_1.$$

In the case $U = \mathbb{I}$ and $\varphi_k = \varphi$ $(k \in \mathbb{N}^*)$ we have

$$Tf = f \star \varphi,$$

and this justifies the notion. We will apply this lemma for operators defined by the sequences

$$\varphi_j^{\langle 1 \rangle} := \sum_{k=j}^{\infty} 2^{-k} \Delta_j D_{2^k}, \quad \varphi_j^{\langle 2 \rangle} := 2^{-j} \sum_{k=0}^{j} \Delta_k D_{2^k} \quad (j \in \mathbb{N}),$$

where

$$\Delta_k g := \sum_{j=0}^{k} 2^{j-1} \tau_{e_j} g \quad (k \in \mathbb{N}).$$

Since

$$\|D_{2^k}\|_1 = 1, \quad \|\Delta_j D_{2^k}\|_1 < 2^j \quad (j, k \in \mathbb{N}),$$

we obtain

$$\|\varphi_j^{\langle 1 \rangle}\|_1 \leq 2^j \sum_{k=j}^{\infty} 2^{-k} = 2, \quad \|\varphi_j^{\langle 2 \rangle}\|_1 = 2^{-j} \sum_{k=0}^{j} 2^k < 2 \quad (j \in \mathbb{N}).$$

Thus Lemma 1 can be applied for every subsequence of these sequences. Let $|J_k| = 2^{-\nu_k}$ denote the lenght of J_k and let us introduce the generalized convolution operators

$$T^{\langle i \rangle} f = \sum_{k=1}^{\infty} (\chi_{J_k} f) \star \varphi_{\nu_k}^{\langle i \rangle} \quad (f \in L^1,\ i = 1, 2). \tag{5.4}$$

Applying Lemma 1, we get

Corollary 5. *The operators* $T^{\langle i \rangle}$ $(i = 1, 2)$ *satisfy*

$$\|T^{\langle i \rangle} f\|_1 \leq 2\|\chi_U f\|_1 \quad (f \in L^1,\ i = 1, 2).$$

Taking (2.4), i.e. the following form of the operators $V_n^{(2)}$

$$(V_n^{(2)} f)(x) = 2^{-n/2} \left\| \sum_{k=0}^{n-1} 2^k \chi_{[0,2^{-k})} \tau_{e_k \dot{+} x} E_n f \right\|_2 ,$$

and applying the decomposition of f introduced in (5.1) we show that the operators $V_n^{(2)}$ can be estimated by the operators $T^{\langle i \rangle}$ on the complementary set $\overline{U} := \mathbb{I} \setminus U$ of U. More precisely we prove

Lemma 2. *Let $g = \sum_{k=1}^{\infty} f_k$, where the f_k's ($k \in \mathbb{N}^*$) are the functions in the Calderon–Zygmund decomposition of f corresponding to the parameter $y > 0$. Denote $|J_k| = 2^{-\nu_k}$ ($k \in \mathbb{N}^*$) the lenght of J_k. Then the following estimation holds at every point $x \in \overline{U}$:*

$$|(V^{(2)} g)(x)| \le 8y \Big((T^{\langle 1 \rangle} |g|)(x) + 4 (T^{\langle 2 \rangle} |g|)(x) \Big) \quad (x \in \overline{U}). \tag{5.5}$$

Proof. If $\nu_j \ge n$ then $E_n f_j = 0$. Therefore, the square of $V^{(2)} g$ can be written in the form

$$\begin{aligned} |(V_n^{(2)} g)(x)|^2 &= 2^{-n} \int_0^1 \left| \sum_{k=0}^{n-1} 2^k \chi_{[0,2^{-k})}(u) \bigg(E_n \Big(\sum_{j:\nu_j < n} f_j \Big) \bigg) (x \dot{+} e_k \dot{+} u) \right|^2 du \\ &= \sum_{(j,k) \in A^{(n)}} \alpha_{(j,k)}^{(n)}(x), \end{aligned}$$

where

$$A^{(n)} := \{ (j,k) : j = (j_1, j_2), k = (k_1, k_2), 0 \le \nu_{j_1}, \nu_{j_2} < n, 0 \le k_1, k_2 < n \} ,$$

and for $(j,k) \in A^{(n)}$ the α's are defined by

$$\begin{aligned} \alpha_{(j,k)}^{(n)}(x) := 2^{-n+k_1+k_2} \int_0^1 \chi_{[0,2^{-k_1 \vee k_2})}(u) (E_n f_{j_1}) \\ \times (x \dot{+} e_{k_1} \dot{+} u)(E_n f_{j_2})(x \dot{+} e_{k_2} \dot{+} u) \, du. \end{aligned} \tag{5.6}$$

Since

$$\alpha_{(\ell,k)}^{(n)}(x) = \alpha_{(\hat{\ell},\hat{k})}^{(n)}(x) \quad \Big(\hat{\ell} := (\ell_2, \ell_1), \hat{k} := (k_2, k_1), \ell = (\ell_1, \ell_2), k = (k_1, k_2) \Big) ,$$

we have that the last sum can be estimated as

$$|(V_n^{(2)}g)(x)|^2 \leq 2 \sum_{(j,k)\in A_1^{(n)}} |\alpha_{(j,k)}^{(n)}(x)|, \tag{5.7}$$

where $A_1^{(n)} := \{(j,k) \in A^{(n)} : \nu_{j_1} \leq \nu_{j_2}\}$. For $\nu_\ell < n$ it follows from (5.1) iii) and iv) that

$$\begin{gathered}(E_n f_\ell)(s) = 0 \quad (s \notin J_\ell),\\ 2^{-n}|(E_n f_\ell)(s)| \leq \int_{J\ell} |f(t)|\,dt \leq 4y|J_\ell| \quad (s \in J_\ell).\end{gathered} \tag{5.8}$$

For $\ell \in \mathbb{N}^*$ and for every index (j,k) set

$$h_\ell(s) := \begin{cases} 0 & (s \notin J_\ell),\\ |J_\ell| & (s \in J_\ell),\end{cases} \tag{5.9}$$

and

$$\begin{gathered}\alpha_{(j,k)}(x) := 2^{k_1+k_2} \int_0^1 \chi_{[0,2^{-k_1\vee k_2})}\\ \times(u \dotplus x)|f_{j_1}(u \dotplus e_{k_1})|h_{j_2}(u \dotplus e_{k_2})\,du.\end{gathered} \tag{5.10}$$

Observe that these functions do not depend on n. Then by (5.6), (5.8) and (5.9) we have

$$\left|\alpha_{(j,k)}^{(n)}(x)\right| \leq 4y\alpha_{(j,k)}(x) \quad \left((j,k) \in A_1^{(n)}\right). \tag{5.11}$$

If $k_i \geq \nu_{j_i}$ then $u \dotplus e_{k_i} \in J_{j_i}$ if and only if $u \in J_{j_i}$. Consequently,

$$\chi_{[0,2^{-k_1\vee k_2})}(u \dotplus x)\chi_{J_{j_i}}(u \dotplus e_{k_i}) = 0 \quad (x \in \overline{U},\ i = 1,2).$$

Hence

$$\alpha_{(j,k)}^{(n)}(x) = 0,\ \text{if either } k_1 \geq \nu_{j_1},\ \text{or } k_2 \geq \nu_{j_2}.$$

for every $x \in \overline{U}$. Thus in the points $x \in \overline{U}$ we have

$$\sum_{(j,k)\in A_1^{(n)}} \left|\alpha_{(j,k)}^{(n)}(x)\right| \leq 4y \sum_{(j,k)\in A} \alpha_{(j,k)}(x) \quad (x \in \overline{U}),$$

where

$$A := \{(j,k) : j \in \mathbb{N}^* \times \mathbb{N}^*,\ k \in \mathbb{N} \times \mathbb{N},\ \nu_{j_1} \le \nu_{j_2},\ k_1 < \nu_{j_1},\ k_2 < \nu_{j_2}\}.$$

The last sum does not depend on n therefore by (5.7) and (5.11) we have that the square of the maximal operator $V^{(2)}$ can be estimated by

$$|(V^{(2)}g)(x)|^2 \le 8y \sum_{(j,k)\in A} \alpha_{(j,k)}(x) \quad (x \in \overline{U}). \tag{5.12}$$

We will decompose the sum according to the following pairwise disjoint subsets of A:

$$\begin{aligned} A &= \{(j,k) \in A : k_1 \le k_2\} \cup \{(j,k) \in A : k_1 > k_2\} \\ &= \{(j,k) \in A : k_1 \le k_2\} \cup A_3 = \{(j,k) \in A : k_1 \le k_2, \nu_{j_1} \le k_2\} \\ &\quad \cup \{(j,k) \in A : k_1 \le k_2, \nu_{j_1} > k_2\} \cup A_3 = A_1 \cup A_2 \cup A_3. \end{aligned}$$

The corresponding sums are

$$F_i(x) := \sum_{(j,k)\in A_i} \alpha_{(j,k)}(x) \quad (x \in \overline{U},\ i = 1,2,3). \tag{5.13}$$

If $(j,k) \in A_1$, then $0 \le k_1 < \nu_{j_1} \le k_2 < \nu_{j_2}$. By (5.9) we have

$$2^{k_2}\chi_{[0,2^{-k_2})}(u \dotplus x) \sum_{j_2 : k_2 < \nu_{j_2}} h_{j_2}(u \dotplus e_{k_2}) \le \frac{1}{2}\chi_{[0,2^{-k_2})}(u \dotplus x).$$

Then it follows from the definition of Δ_ℓ and from $T^{\langle 1\rangle}$ and by (5.10) that

$$F_1(x) \le \sum_{j_1=1}^{\infty} \sum_{k_2 \ge \nu_{j_1}} \sum_{k_1=0}^{\nu_{j_1}} 2^{k_1-1} \int_0^1 \chi_{[0,2^{-k_2})}(u \dotplus x)|f_{j_1}(u \dotplus e_{k_1})|\, du$$

$$= \sum_{j_1=1}^{\infty} \left(|f_{j_1}| \star \Delta_{\nu_{j_1}} \left(\sum_{k_2=\nu_{j_1}}^{\infty} 2^{-k_2} D_{2^{k_2}} \right) \right)(x) = (T^{\langle 1\rangle}|g|)(x) \quad (x \in \overline{U}).$$

If $(j,k) \in A_2$, then $0 \le k_1 \le k_2 < \nu_{j_1} \le \nu_{j_2}$. Again by (5.9) we have

$$\sum_{j_2 : \nu_{j_1} \le \nu_{j_2}} h_{j_2}(u \dotplus e_{k_2}) \le 2^{-\nu_{j_1}} \quad (u \in \mathbb{I}). \tag{5.14}$$

Hence it follows in a similar way as before that

$$F_2(x) \le \sum_{j_1=1}^{\infty} \sum_{k_2=0}^{\nu_{j_1}} \sum_{k_1=0}^{k_2} 2^{k_1-\nu_{j_1}} \int_0^1 D_{2^{k_2}}(u \dotplus x \dotplus e_{k_1})|f_{j_1}(u)|\,du$$

$$= 2\sum_{j_1=1}^{\infty} \left(|f_{j_1}| \star \left(2^{-\nu_{j_1}} \sum_{k_2=0}^{\nu_{j_1}} \Delta_{k_2} D_{2^{k_2}}\right)\right)(x) = 2(T^{\langle 2\rangle}|g|)(x) \quad (x \in \overline{U}).$$

Finally let $(j,k) \in A_3$. Then $0 \le k_2 < k_1 < \nu_{j_1} \le \nu_{j_2}$, therefore by (5.14) we have

$$F_3(x) \le \sum_{j_1=1}^{\infty} 2^{-\nu_{j_1}} \sum_{k_1=0}^{\nu_{j_1}} \sum_{k_2=0}^{k_1-1} 2^{k_2} \int_0^1 D_{2^{k_1}}(u \dotplus x)|f_{j_1}(u \dotplus e_{k_1}|\,du$$

$$= \sum_{j_1=1}^{\infty} 2^{-\nu_{j_1}} \sum_{k_1=0}^{\nu_{j_1}} \sum_{k_2=0}^{k_1-1} 2^{k_2} \int_0^1 D_{2^{k_1}}(u \dotplus x)|f_{j_1}(u)|\,du$$

$$= \sum_{j_1=1}^{\infty} 2^{-\nu_{j_1}} \left(|f_{j_1}| \star \left(\sum_{k_1=0}^{\nu_{j_1}} 2^{k_1} D_{2^{k_1}}\right)\right)(x) \quad (x \in \overline{U}).$$

Recall the definition of Δ_ℓ to see

$$2^{\ell-1} D_{2^\ell} \le \Delta_\ell D_{2^\ell}.$$

Consequently $F_3(x) \le 2(T^{\langle 2\rangle}|g|)(x)$ holds true in the points of $\overline{U}$.

Summarizing our inequalities we have by (5.12) and (5.13) that

$$|(V^{(2)}g)(x)|^2 \le 8y(F_1(x) + F_2(x) + F_3(x))$$
$$\le 8y\left((T^{\langle 1\rangle}|g|)(x) + 4(T^{\langle 2\rangle}|g|)(x)\right) \quad (x \in \overline{U}).$$

Lemma 2 is proved.

Now we prove Theorem 2.

Proof of Therem 2. Let us take the representation (2.4) of the operators $V_n^{(2)}$ and apply the inequality $\|E_n f\|_\infty \le \|f\|_\infty$ to obtain

$$(V_n^{(2)} f)(x) \le 2^{-n/2}\|f\|_\infty \left\|\sum_{k=0}^{n-1} 2^k \chi_{[0,2^{-k})}\right\|_2 \le 2\|f\|_\infty.$$

Taking the supremum with respect to n we get proof of part i) of our theorem.

In the proof of part ii) we start with the number $y > \|f\|_1$ and apply the Calderon–Zygmund decomposition for f. With the notations of this decomposition lemma the function f can be written as $f = f_0 + g$, where $\|f_0\|_\infty \le 2y$. Applying the inequality of part i) and the subadditivity of $V^{(2)}$ we get

$$(V^{(2)}f)(x) \le (V^{(2)}f_0)(x) + (V^{(2)}g)(x) \le 4y + (V^{(2)}g)(x)$$
$$\text{(for a.e. } x \in \mathbb{I}).$$

Hence

$$\left|\left\{x : (V^{(2)}f)(x) > 5y\right|\right\} \le \left|\left\{x : (V^{(2)}g)(x) > y\right|\right\}. \tag{5.15}$$

By (5.1) v), vi) we have

$$\left|\left\{x \in U : (V^{(2)}g)(x) > y\right|\right\} \le |U| \le \frac{1}{y}\int_U |f(s)|\, ds, \tag{5.16}$$

therefore it is enough to estimate the function $V^{(2)}g$ in the points of $\overline{U}$. By Lemma 2 we have

$$\begin{aligned}\left|\left\{x \in \overline{U} : (V^{(2)}g)(x) > y\right|\right\} &\le \frac{1}{y^2}\int_{\overline{U}} |(V^{(2)}g)(x)|^2\, dx \\ &\le \frac{8}{y}\int_{\overline{U}} \Big((T^{\langle 1\rangle}g)(x) + 4(T^{\langle 2\rangle}g)(x)\Big)\, dx.\end{aligned}$$

Applying Corollary 5 we get

$$\left|\left\{x \in \overline{U} : (V^{(2)}g)(x) > y\right|\right\} \le \frac{80}{y}\int_U |g(s)|\, ds. \tag{5.17}$$

On the basis of (5,1) ii) iv), v) and vi) we have

$$\int_U |g(s)|\, ds = \sum_{j=1}^{\infty}\int_{J_j} |f_j(s)|\, ds \le 4y\sum_{j=1}^{\infty}|J_j| \le 4\int_U |f(s)|\, ds.$$

Therefore, it follows from (5.15), (5.16) , (5.17) and (4.1) vi) that (2.9) ii) holds for every $y > \|f\|_1$. Finally, if we apply $(E^*|f|)(x) \ge \|f\|_1$ $(x \in \mathbb{I})$

for the case $\|f\|_1 > y$ we get that the set $\{E^*|f| > y\}$ is equal to the interval $[0,1)$. Consequently, in this case we have $321\|f\|_1/y \geq 321$ on the right hand side wich is greater than the left hand side.

Theorem 2 is proved.

Proof of Corollary 1. Let $F := V^{(2)}f$ and $g := E^*|f|$. Inequality (2.9) is equivalent to

$$\int_0^1 \chi_{\{F>5y\}}(s)\,ds \leq \frac{321}{y}\int_0^1 \chi_{\{g>y\}}(s)|f(s)|\,ds \quad (y>0). \tag{5.18}$$

Let us take the left side. Multiply it by py^{p-1} then integrate with respect to y and apply Fubini's theorem to obtain

$$\int_0^\infty py^{p-1}\left(\int_0^1 \chi_{\{F>5y\}}(s)ds\right)dy$$
$$= \int_0^1 \left(\int_0^{F(s)/5} py^{p-1}\,dy\right)ds = \int_0^1 |F(s)/5|^p\,ds.$$

Applying the same procedure for the right hand side, except for the factor 321, we get

$$\int_0^\infty py^{p-2}\left(\int_0^1 \chi_{\{g>y\}}(s)|f(s)|\,ds\right)dy = \int_0^1 \left(|f(s)|\int_0^{g(s)} py^{p-2}\,dy\right)ds$$
$$= \frac{p}{p-1}\int_0^1 |f(s)|\,|g(s)|^{p-1}ds.$$

Thus we proved

$$\int_0^1 |F(s)|^p\,ds \leq 321\cdot 5^p\frac{p}{p-1}\int_0^1 |f(s)|\,|g(s)|^{p-1}\,ds. \tag{5.19}$$

Applying Hölder's and Doob's inequalities and $(p-1)q = p$, and $p/q+1 = p$ we get

$$\int_0^1 |f(s)|\,|g(s)|^{p-1}\,ds \leq \|f\|_p\,\|g\|_p^{p/q} \leq \left(\frac{p}{p-1}\right)^{p/q}\|f\|_p^p.$$

Comparing this with (5.19) we obtain

$$\|F\|_p \le C\frac{p}{p-1}\|f\|_p \quad (C < 5\cdot 321).$$

Proof of Corollary 2. Applying inequality (5.18) for $y \ge \|f\|_1$ and integrating with respect to y we get

$$\int_{\|f\|_1}^{\infty} |\{F > 5y\}|\, dy \le 321 \int_0^1 |f(s)| \left(\int_{\|f\|_1}^{g(s)} \frac{dy}{y}\right) ds$$
$$= 321 \int_0^1 |f(s)| \log \frac{g(s)}{\|f\|_1}\, ds.$$

Since

$$\int_0^{\|f\|_1} |\{F > 5y\}|\, dy \le \|f\|_1,$$

we can finish the proof by recalling that

$$\frac{1}{5}\int_0^1 F(s)\, ds = \int_0^{\infty} |\{F > 5y\}|\, dy.$$

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(Received November 18, 1997)