

(C, α) means of d -dimensional trigonometric-Fourier series

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*Dedicated to the 60th birthday of
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Abstract. The d -dimensional classical Hardy spaces $H_p(\mathbf{T}^d)$ are introduced and it is shown that the maximal operator of the (C, α) ($\alpha = (\alpha_1, \dots, \alpha_d)$) means of a distribution is bounded from $H_p(\mathbf{T}^d)$ to $L_p(\mathbf{T}^d)$ ($d/(d+1), 1/(\alpha_k+1) < p < \infty$) provided that the supremum in the maximal operator is taken over a positive cone. Moreover, we prove that the (C, α) means are uniformly bounded on the spaces $H_p(\mathbf{T}^d)$ whenever $d/(d+1), 1/(\alpha_k+1) < p < \infty$. Thus, in case $f \in H_p(\mathbf{T})$, the Cesàro means converge to f in $H_p(\mathbf{T}^d)$ norm ($d/(d+1), 1/(\alpha_k+1) < p < \infty$). The same results are proved for the conjugate (C, α) means, too.

1. Introduction

The Hardy–Lorentz spaces $H_{p,q}(\mathbf{T}^d)$ of distributions are introduced with the $L_{p,q}(\mathbf{T}^d)$ Lorentz norms of the non-tangential maximal function. Of course, $H_p(\mathbf{T}^d) = H_{p,p}(\mathbf{T}^d)$ are the usual Hardy spaces ($0 < p \leq \infty$).

For multi-dimensional trigonometric-Fourier series MARCINKIEVICZ and ZYGMUND [7] proved that the Fejér means $\sigma_n^1 f$ of a function $f \in L_1(\mathbf{T}^d)$ converge a.e. to f as $\min(n_1, \dots, n_d) \rightarrow \infty$ provided that n is in

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a positive cone, i.e., provided that $2^{-\tau} \leq n_k/n_j \leq 2^\tau$ for every $k, j = 1, \dots, d$ and for some $\tau \geq 0$ ($n = (n_1, \dots, n_d) \in \mathbf{N}^d$).

Recently the author [15] obtained the same convergence result for the (C, α) means σ_n^α by proving the weak type inequality

$$\sup_{\rho > 0} \rho \lambda(\sigma_*^\alpha f > \rho) \leq C \|f\|_1 \quad (f \in L_1(\mathbf{T}^d))$$

where $\sigma_*^\alpha := \sup_{\substack{2^{-\tau} \leq n_k/n_j \leq 2^\tau \\ k, j=1, \dots, d}} |\sigma_n^\alpha|$, $\alpha = (\alpha_1, \dots, \alpha_d)$ and $0 < \alpha_k \leq 1$.

Moreover, for the Fejér means (if $\alpha_k = 1$) the author [15] verified that σ_*^1 is bounded from $H_{p,q}(\mathbf{T}^d)$ to $L_{p,q}(\mathbf{T}^d)$ if $(2d+1)/(2d+2) < p \leq \infty$ and $0 < q \leq \infty$. Under some conditions on α we proved also a similar result for the (C, α) means. The one-dimensional results are described in WEISZ [14].

In this paper we sharpen and generalize these results for arbitrary $0 < \alpha_k \leq 1$. We will show that the maximal operator σ_*^α is bounded from $H_{p,q}(\mathbf{T}^d)$ to $L_{p,q}(\mathbf{T}^d)$ whenever $d/(d+1), 1/(\alpha_k+1) < p \leq \infty, 0 < q \leq \infty$. We introduce the conjugate distributions (or Riesz transforms) $\tilde{f}^{(i)} = R_i f$, the conjugate (C, α) means $\tilde{\sigma}_n^{(i);\alpha}$ and the conjugate maximal operators $\tilde{\sigma}_*^{(i);\alpha}$ where $i = 0, 1, \dots, d$. We obtain that the operator $\tilde{\sigma}_*^{(i);\alpha}$ is also of type $(H_{p,q}(\mathbf{T}^d), L_{p,q}(\mathbf{T}^d))$ if $d/(d+1), 1/(\alpha_k+1) < p \leq \infty, 0 < q \leq \infty$ and of weak type $(1, 1)$.

A usual density argument implies then that, besides the convergence results mentioned above, the conjugate (C, α) means $\tilde{\sigma}_n^{(i);\alpha} f$ converge a.e. to $\tilde{f}^{(i)}$ as $n \rightarrow \infty$ and $2^{-\tau} \leq n_k/n_j \leq 2^\tau$, provided that $f \in L_1(\mathbf{T}^d)$. Note that $\tilde{f}^{(i)}$ is not necessarily integrable whenever f is.

We will prove also that the operators σ_n^α and $\tilde{\sigma}_n^{(i);\alpha}$ ($n \in \mathbf{N}$) are uniformly bounded from $H_{p,q}(\mathbf{T}^d)$ to $H_{p,q}(\mathbf{T}^d)$ if $d/(d+1), 1/(\alpha_k+1) < p \leq \infty, 0 < q \leq \infty$. From this it follows that $\sigma_n^\alpha f \rightarrow f$ and $\tilde{\sigma}_n^{(i);\alpha} f \rightarrow \tilde{f}^{(i)}$ in $H_{p,q}(\mathbf{T}^d)$ norm as $n \rightarrow \infty, 2^{-\tau} \leq n_k/n_j \leq 2^\tau$, whenever $f \in H_{p,q}(\mathbf{T})$ and $d/(d+1), 1/(\alpha_k+1) < p \leq \infty, 0 < q \leq \infty$.

2. Hardy spaces and Riesz transforms

For a set $\mathbf{X} \neq \emptyset$ let $\mathbf{X}^d$ be its Cartesian product taken with itself d times ($d \in \mathbf{N}$), moreover, let $\mathbf{T} := [-\pi, \pi)$ and λ be the d -dimensional Lebesgue measure. We also use the notation $|I|$ for the Lebesgue measure

of the set I . We briefly write L_p instead of the real $L_p(\mathbf{T}^d, \lambda)$ space while the norm (or quasinorm) of this space is defined by $\|f\|_p := (\int_{\mathbf{T}^d} |f|^p d\lambda)^{1/p}$ ($0 < p \leq \infty$). For simplicity, we assume that for a function $f \in L_1$ we have $\int_{\mathbf{T}^d} f d\lambda = 0$.

The distribution function of a Lebesgue-measurable function f is defined by

$$\lambda(\{|f| > \rho\}) := \lambda(\{x : |f(x)| > \rho\}) \quad (\rho \geq 0).$$

The *weak* L_p space L_p^* ($0 < p < \infty$) consists of all measurable functions f for which

$$\|f\|_{L_p^*} := \sup_{\rho > 0} \rho \lambda(\{|f| > \rho\})^{1/p} < \infty$$

while we set $L_\infty^* = L_\infty$.

The spaces L_p^* are special cases of the more general Lorentz spaces $L_{p,q}$. In their definition another concept is used. For a measurable function f the *non-increasing rearrangement* is defined by

$$\check{f}(t) := \inf\{\rho : \lambda(\{|f| > \rho\}) \leq t\}.$$

Lorentz space $L_{p,q}$ is defined as follows: for $0 < p < \infty, 0 < q < \infty$

$$\|f\|_{p,q} := \left(\int_0^\infty \check{f}(t)^q t^{q/p} \frac{dt}{t} \right)^{1/q}$$

while for $0 < p \leq \infty$

$$\|f\|_{p,\infty} := \sup_{t>0} t^{1/p} \check{f}(t).$$

Let

$$L_{p,q} := L_{p,q}(\mathbf{T}^d, \lambda) := \{f : \|f\|_{p,q} < \infty\}.$$

One can show the following equalities:

$$L_{p,p} = L_p, \quad L_{p,\infty} = L_p^* \quad (0 < p \leq \infty)$$

(see e.g. BENNETT, SHARPLEY [1] or BERGH, LÖFSTRÖM [2]).

We introduce the $H_p(\mathbf{T}^d)$ Hardy space in a similar way as in WEISZ [15]. Let us fix $d \geq 1$. For $n = (n_1, \dots, n_d) \in \mathbf{Z}^d$ and $x = (x_1, \dots, x_d) \in \mathbf{T}^d$ set $n \cdot x := \sum_{k=1}^d n_k x_k$ and $|n| := (\sum_{k=1}^d |n_k|^2)^{1/2}$. Let f be a distribution on $C^\infty(\mathbf{T}^d)$ (briefly $f \in \mathcal{D}'(\mathbf{T}^d) = \mathcal{D}'$). The n th Fourier coefficient

is defined by $\hat{f}(n) := f(e^{-in \cdot x})$ where $\imath = \sqrt{-1}$ and $n \in \mathbf{Z}^d$. In special case, if f is an integrable function then

$$\hat{f}(n) = \frac{1}{(2\pi)^d} \int_{\mathbf{T}^d} f(x) e^{-in \cdot x} dx.$$

For simplicity, we assume that, for a distribution $f \in \mathcal{D}'$, we have $\hat{f}(0) = 0$.

For $f \in \mathcal{D}'$ and $t > 0$ define the harmonic function u by

$$u(x, t) := (f * P_t)(x)$$

where $*$ denotes the convolution and

$$P_t(x) := \sum_{k \in \mathbf{Z}^d} e^{-t|k|} e^{\imath k \cdot x} \quad (x \in \mathbf{T}^d)$$

is the Poisson kernel. Let $\Gamma := \{(x, t) : |x| < t\}$, a cone whose vertex is the origin. We denote by $\Gamma'(x)$ ($x \in \mathbf{T}^d$) the translate of Γ so that its vertex is x . Set

$$\Gamma(x) = \bigcup_{k \in \mathbf{Z}^d} \Gamma'((x_i + k_i 2\pi)) \cap \mathbf{T}^d.$$

The non-tangential maximal function is defined by

$$u^*(x) := \sup_{(x', t) \in \Gamma(x)} |u(x', t)| \quad (\alpha > 0).$$

For $0 < p, q \leq \infty$ the *Hardy-Lorentz space* $H_{p,q}(\mathbf{T}^d) = H_{p,q}$ consists of all distributions f for which $u^* \in L_{p,q}$ and set

$$\|f\|_{H_{p,q}} := \|u^*\|_{p,q}.$$

Note that in case $p = q$ the usual definition of Hardy spaces $H_{p,p} = H_p$ are obtained. It is known that if $f \in H_p$ ($0 < p < \infty$) then $f(x) = \lim_{t \rightarrow 0} u(x, t)$ in the sense of distributions (see FEFFERMAN, STEIN [6]). Recall that $L_1 \subset H_{1,\infty}$, more exactly,

$$(1) \quad \|f\|_{H_{1,\infty}} = \sup_{\rho > 0} \rho \lambda(u^* > \rho) \leq C \|f\|_1 \quad (f \in L_1).$$

Moreover, $H_{p,q} \sim L_{p,q}$ for $1 < p < \infty$, $0 < q \leq \infty$ (see FEFFERMAN, STEIN [6], STEIN [10], FEFFERMAN, RIVIERE, SAGHER [5]).

The following interpolation result concerning Hardy-Lorentz spaces will be used several times in this paper (see FEFFERMAN, RIVIERE, SAGHER [5] and also WEISZ [16]).

Theorem A. *If a sublinear (resp. linear) operator T is bounded from H_{p_0} to L_{p_0} (resp. to H_{p_0}) and from L_{p_1} to L_{p_1} ($p_0 \leq 1 < p_1 \leq \infty$) then it is also bounded from $H_{p,q}$ to $L_{p,q}$ (resp. to $H_{p,q}$) if $p_0 < p < p_1$ and $0 < q \leq \infty$.*

For a distribution

$$f \sim \sum_{n \in \mathbf{Z}^d} \hat{f}(n) e^{in \cdot x}$$

the *Riesz transforms* or the *conjugate distributions* are defined by

$$\tilde{f}^{(i)} := R_i f \sim \sum_{n \in \mathbf{Z}^d} -i \frac{n_i}{|n|} \hat{f}(n) e^{in \cdot x} \quad (i = 1, \dots, d).$$

We use the notation $\tilde{f}^{(0)} := f$.

As is well known, if f is an integrable function then the conjugate functions $\tilde{f}^{(i)}$ ($i = 1, \dots, d$) do exist almost everywhere, but they are not integrable in general.

FEFFERMAN and STEIN [6] and UCHIYAMA [13] verified that if $f \in H_p$ ($0 < p < \infty$) then all conjugate distributions are also in H_p and

$$(2) \quad \|\tilde{f}^{(i)}\|_{H_p} \leq C_p \|f\|_{H_p} \quad (i = 1, \dots, d).$$

Furthermore, if $(d-1)/d < p < \infty$ then the following equivalence holds:

$$(3) \quad \|f\|_{H_p} \sim \|f\|_p + \sum_{i=1}^d \|\tilde{f}^{(i)}\|_p.$$

3. (C, α) summability of d -dimensional trigonometric-Fourier series

Denote by $s_n f$ and $\tilde{s}_n^{(i)} f$ the n th partial sum and conjugate partial sum of the Fourier series of a distribution f , respectively, namely,

$$s_n f(x) := \sum_{j=1}^d \sum_{k_j=-n_j}^{n_j} \hat{f}(k) e^{ik \cdot x}$$

and

$$\tilde{s}_n^{(i)} f(x) := \sum_{j=1}^d \sum_{k_j=-n_j}^{n_j} -i \frac{k_i}{|k|} \hat{f}(k) e^{ik \cdot x}.$$

Let $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbf{N}^d$ with $0 < \alpha_k \leq 1$ ($k = 1, \dots, d$) and let

$$A_j^\gamma := \binom{j+\gamma}{j} = \frac{(\gamma+1)(\gamma+2)\dots(\gamma+j)}{j!} = O(j^\gamma) \quad (j \in \mathbf{N}, 0 < \gamma \leq 1)$$

(see ZYGMUND [17]). The (C, α) means of a distribution f are defined by

$$\begin{aligned} \sigma_n^\alpha f &:= \prod_{i=1}^d \frac{1}{A_{n_i}^{\alpha_i}} \sum_{i=1}^d \sum_{k_i=0}^{n_i} A_{n_i-k_i}^{\alpha_i-1} s_{k_i} f \\ &= \prod_{i=1}^d \frac{1}{A_{n_i}^{\alpha_i}} \sum_{i=1}^d \sum_{k_i=-n_i}^{n_i} A_{n_i-|k_i|}^{\alpha_i} \hat{f}(k) e^{ik \cdot x} = f * (K_{n_1}^{\alpha_1} \times \dots \times K_{n_d}^{\alpha_d}) \end{aligned}$$

where the K_j^γ kernel satisfies the conditions

$$(4) \quad |K_j^\gamma(t)| \leq C_\gamma j \quad (0 < |t| < \pi)$$

and

$$(5) \quad |K_j^\gamma(t)| \leq \frac{C_\gamma}{j^\gamma |t|^{\gamma+1}} \quad (0 < |t| < \pi)$$

for $j \in \mathbf{N}$ and $0 < \gamma \leq 1$. From this it follows that

$$(6) \quad |K_j^\gamma(t)| \leq \frac{C_\gamma j^{\eta+\gamma(\eta-1)}}{|t|^{(\gamma+1)(1-\eta)}} \quad (0 < |t| < \pi)$$

where $0 \leq \eta \leq 1$ is arbitrary. We have similar estimates for the derivative of the kernel:

$$(7) \quad |(K_j^\gamma)'(t)| \leq C_\gamma j^2 \quad (0 < |t| < \pi)$$

and

$$(8) \quad |(K_j^\gamma)'(t)| \leq \frac{C_\gamma}{j^{\gamma-1} |t|^{\gamma+1}} \quad (0 < |t| < \pi)$$

for $j \in \mathbf{N}$ and $0 < \gamma \leq 1$ (see ZYGMUND [17], Vol. II, p. 60). Thus, for $0 \leq \theta \leq 1$,

$$(9) \quad |(K_j^\gamma)'(t)| \leq \frac{C_\gamma j^{\theta+1+\gamma(\theta-1)}}{|t|^{(\gamma+1)(1-\theta)}} \quad (0 < |t| < \pi).$$

It is easy to see that (4) and (5) imply

$$(10) \quad \int_{\mathbf{T}} |K_j^\gamma| d\lambda \leq C_\gamma \quad (j \in \mathbf{N}).$$

The *conjugate* (C, α) *means* of a distribution f are introduced by

$$\begin{aligned} \tilde{\sigma}_n^{(i);\alpha} f &:= \prod_{i=1}^d \frac{1}{A_{n_i}^{\alpha_i}} \sum_{i=1}^d \sum_{k_i=0}^{n_i} A_{n_i-k_i}^{\alpha_i-1} \tilde{s}_k^{(i)} f \\ &= \prod_{i=1}^d \frac{1}{A_{n_i}^{\alpha_i}} \sum_{i=1}^d \sum_{k_i=-n_i}^{n_i} A_{n_i-|k_i|}^{\alpha_i} (-i)^{\frac{k_i}{|k|}} \hat{f}(k) e^{ik \cdot x} \\ &= \tilde{f}^{(i)} * (K_{n_1}^{\alpha_1} \times \dots \times K_{n_d}^{\alpha_d}). \end{aligned}$$

For a fixed $\tau \geq 0$ the *restricted maximal* and *restricted maximal conjugate* (C, α) *operators* are defined by

$$\sigma_*^\alpha f := \sup_{\substack{2^{-\tau} \leq n_k/n_j \leq 2^\tau \\ k,j=1,\dots,d}} |\sigma_n^\alpha f|$$

and

$$\tilde{\sigma}_*^{(i);\alpha} f := \sup_{\substack{2^{-\tau} \leq n_k/n_j \leq 2^\tau \\ k,j=1,\dots,d}} |\tilde{\sigma}_n^{(i);\alpha} f|.$$

Obviously,

$$(11) \quad \tilde{\sigma}_n^{(i);\alpha} f = \sigma_n^\alpha \tilde{f}^{(i)} \quad \text{and} \quad \tilde{\sigma}_*^{(i);\alpha} f = \sigma_*^\alpha \tilde{f}^{(i)} \quad (i = 0, 1, \dots, d).$$

4. The boundedness of the maximal (C, α) operator

A *generalized interval* on $\mathbf{T}$ is either an interval $I \subset \mathbf{T}$ or $I = [-\pi, x) \cup [y, \pi)$. A *generalized cube* on $\mathbf{T}^d$ is the Cartesian product $I_1 \times \dots \times I_d$ of d generalized intervals with $|I_1| = \dots = |I_d|$. A bounded measurable function a is a *p-atom* if there exists a generalized cube R such that

- (i) $\int_R a(x) x^\beta dx = 0$ for all multi-indices $\beta = (\beta_1, \dots, \beta_d) \in \mathbf{N}^d$ with $|\beta| \leq [d(1/p - 1)]$, the integer part of $d(1/p - 1)$,
- (ii) $\|a\|_\infty \leq |R|^{-1/p}$,
- (iii) $\{a \neq 0\} \subset R$.

If I is a generalized interval then let $4I$ be the generalized interval with the same center as I and with length $4|I|$. For a generalized cube $R = I_1 \times \dots \times I_d$ let $4R = 4I_1 \times \dots \times 4I_d$.

An operator T which maps the set of distributions into the collection of measurable functions, will be called *p-quasi-local* if there exists a constant $C_p > 0$ such that

$$\int_{\mathbf{T}^d \setminus 4R} |Ta|^p d\lambda \leq C_p$$

for every p-atom a where R is the support of the atom. The following result can be found in WEISZ [15]:

Theorem B. *Suppose that the operator T is sublinear and p-quasi-local for some $0 < p \leq 1$. If T is bounded from L_{p_1} to L_{p_1} for a fixed $1 < p_1 \leq \infty$ then*

$$\|Tf\|_p \leq C_p \|f\|_{H_p} \quad (f \in H_p).$$

Now we can formulate our main result.

Theorem 1. *Suppose that $\max\{d/(d+1), 1/(\alpha_k+1), k=1, \dots, d\} =: p_0 < p < \infty$, $0 < q \leq \infty$ and $0 < \alpha_k \leq 1$ ($k=1, \dots, d$). Then*

$$(12) \quad \|\sigma_*^\alpha f\|_{p,q} \leq C_{p,q} \|f\|_{H_{p,q}} \quad (f \in H_{p,q}).$$

PROOF. For simplicity we prove the result for $d=2$, only. For $d>2$ the verification is very similar. Now we denote the elements of $\mathbf{N}^2$ by (n, m) and we write (α, β) instead of (α_1, α_2) .

By Theorems A and B the proof of Theorem 1 will be complete if we show that the operator $\sigma_*^{\alpha, \beta}$ is p-quasi-local for each $p_0 < p \leq 1$ and is bounded from L_∞ to L_∞ .

The boundedness follows from (10). Let a be an arbitrary p-atom with support $R = I \times J$ and $2^{-K-1} < |I|/\pi = |J|/\pi \leq 2^{-K}$ ($K \in \mathbf{N}$). We can suppose that the center of R is zero. In this case

$$[-\pi 2^{-K-2}, \pi 2^{-K-2}] \subset I, J \subset [-\pi 2^{-K-1}, \pi 2^{-K-1}].$$

Choose $r \in \mathbf{N}$ such that $r-1 < \tau \leq r$. It is easy to see that if $n \geq k$ or $m \geq k$ for a fixed $k \in \mathbf{N}$ then we have $n, m \geq k 2^{-r}$. Indeed, since (n, m) is in a cone, $n \geq k$ implies

$$m \geq 2^{-\tau} n \geq k 2^{-r}.$$

To prove the quasi-locality of the operator $\sigma_*^{\alpha,\beta}$ we have to integrate $|\sigma_*^{\alpha,\beta} a|^p$ over $\mathbf{T}^2 \setminus 4R$. We do this in three steps.

Step 1. Integrating over $(\mathbf{T} \setminus 4I) \times 4J$. Obviously,

$$\begin{aligned}
 (13) \quad & \int_{\mathbf{T} \setminus 4I} \int_{4J} |\sigma_*^{\alpha,\beta} a(x, y)|^p dx dy \\
 & \leq \sum_{|i|=1}^{2^K-1} \int_{\pi i 2^{-K}}^{\pi(i+1)2^{-K}} \int_{4J} |\sigma_*^{\alpha,\beta} a(x, y)|^p dx dy \\
 & \leq \sum_{|i|=1}^{2^K-1} \int_{\pi i 2^{-K}}^{\pi(i+1)2^{-K}} \int_{4J} \sup_{n, m \geq r_i 2^{-r}} |\sigma_{n,m}^{\alpha,\beta} a(x, y)|^p dx dy \\
 & \quad + \sum_{|i|=1}^{2^K-1} \int_{\pi i 2^{-K}}^{\pi(i+1)2^{-K}} \int_{4J} \sup_{n, m < r_i} |\sigma_{n,m}^{\alpha,\beta} a(x, y)|^p dx dy \\
 & = (A) + (B)
 \end{aligned}$$

where $r_i := \left\lceil \frac{2^K}{|i|^\delta} \right\rceil$ ($i \in \mathbf{N}$) with $\delta > 0$ chosen later. We can suppose that $i > 0$.

The term (A) was estimated for $\alpha = \beta = 1$ in WEISZ [15]. For the sake of the completeness we give the details in the general case. Using (5), (10) and the definition of the atom we conclude

$$\begin{aligned}
 |\sigma_{n,m}^{\alpha,\beta} a(x, y)| &= \frac{1}{(2\pi)^2} \left| \int_I \int_J a(t, u) K_n^\alpha(x-t) K_m^\beta(y-u) dt du \right| \\
 &\leq C_p 2^{2K/p} \int_I \frac{1}{n^\alpha |x-t|^{\alpha+1}} dt.
 \end{aligned}$$

If $x \in [\pi i 2^{-K}, \pi(i+1)2^{-K})$ ($i \geq 1$) and $t \in I$ then

$$(14) \quad \frac{1}{|x-t|^\nu} \leq \frac{1}{(\pi i 2^{-K} - \pi 2^{-K-1})^\nu} \leq \frac{C 2^{K\nu}}{i^\nu} \quad (\nu > 0).$$

Hence

$$|\sigma_{n,m}^{\alpha,\beta} a(x, y)| \leq C_p 2^{2K/p+K\alpha} \frac{1}{n^\alpha i^{\alpha+1}}.$$

Since $n \geq r_i 2^{-r}$, we obtain

$$(A) \leq C_p \sum_{i=1}^{2^K-1} 2^{-2K} 2^{2K+K\alpha p} \frac{1}{r_i^{\alpha p} i^{(\alpha+1)p}} \leq C_p \sum_{i=1}^{2^K-1} \frac{1}{i^{(\alpha+1)p-\alpha\delta p}}$$

which is a convergent series if

$$(15) \quad \delta < \frac{(\alpha+1)p-1}{\alpha p}.$$

Now we consider (B). Let

$$A_1(x, u) := \int_{-\pi}^x a(t, u) dt \quad (x, u \in \mathbf{T})$$

and

$$A(x, y) := \int_{-\pi}^y A_1(x, u) du \quad (x, y \in \mathbf{T}).$$

Observe that

$$(16) \quad |A_1(x, u)| \leq |I|^{1-2/p}, \quad |A(x, y)| \leq |I|^{2-2/p}.$$

Integrating by parts we can see that

$$(17) \quad \begin{aligned} & \int_I a(t, u) K_n^\alpha(x-t) dt \\ &= \left[A_1(t, u) K_n^\alpha(x-t) \right]_{-\mu}^\mu - \int_I A_1(t, u) (K_n^\alpha)'(x-t) dt \\ &= A_1(\mu, u) K_n^\alpha(x-\mu) - \int_I A_1(t, u) (K_n^\alpha)'(x-t) dt \end{aligned}$$

where $I = J = [-\mu, \mu]$. By (4), (5), (16) and (14),

$$\begin{aligned} & \left| \int_J A_1(\mu, u) K_n^\alpha(x-\mu) K_m^\beta(y-u) du \right| \\ & \leq C_p 2^{2K/p-K} 2^{-K} \frac{1}{n^\alpha |x-\mu|^{\alpha+1}} m \leq C_p 2^{2K/p-2K} n^{1-\alpha} \frac{2^{K(\alpha+1)}}{i^{\alpha+1}} \\ & = C_p 2^{2K/p+K\alpha-K} n^{1-\alpha} \frac{1}{i^{\alpha+1}} \end{aligned}$$

whenever $x \in [\pi i 2^{-K}, \pi(i+1)2^{-K})$.

On the other hand, by (5), (10), (16) and (14),

$$\begin{aligned} & \left| \int_J \int_I A_1(t, u) (K_n^\alpha)'(x-t) K_m^\beta(y-u) du dt \right| \\ & \leq C_p 2^{2K/p-K} \int_I \frac{1}{n^{\alpha-1} |x-t|^{\alpha+1}} dt \leq C_p 2^{2K/p+K\alpha-K} n^{1-\alpha} \frac{1}{i^{\alpha+1}} \end{aligned}$$

in case $x \in [\pi i 2^{-K}, \pi(i+1)2^{-K})$. The inequality $n < r_i$ imply

$$\begin{aligned} (B) & \leq C_p \sum_{i=1}^{2^K-1} 2^{-2K} 2^{2K+K\alpha p-Kp} r_i^{(1-\alpha)p} \frac{1}{i^{(\alpha+1)p}} \\ & \leq C_p \sum_{i=1}^{2^K-1} \frac{1}{i^{(\alpha+1)p+(1-\alpha)\delta p}} \end{aligned}$$

which is independent of K if

$$\delta > \frac{1 - (\alpha + 1)p}{(1 - \alpha)p}.$$

This together with (15) yields that $p > 1/(\alpha + 1)$. Hence we have proved that

$$(18) \quad \int_{\mathbf{T} \setminus 4I} \int_{4J} |\sigma_*^{\alpha, \beta} a(x, y)|^p dx dy \leq C_p$$

for $p > 1/(\alpha + 1)$ where C_p depends only on p , τ , α and β .

Step 2. Integrating over $(\mathbf{T} \setminus 4I) \times (\mathbf{T} \setminus 4J)$. Similarly to (13),

$$\begin{aligned} & \int_{\mathbf{T} \setminus 4I} \int_{\mathbf{T} \setminus 4J} |\sigma_*^{\alpha, \beta} a(x, y)|^p dx dy \\ & \leq \sum_{|i|=1}^{2^K-1} \sum_{|j|=1}^{2^K-1} \int_{\pi i 2^{-K}}^{\pi(i+1)2^{-K}} \int_{\pi j 2^{-K}}^{\pi(j+1)2^{-K}} \sup_{n, m \geq r_{i,j} 2^{-r}} |\sigma_{n,m}^{\alpha, \beta} a(x, y)|^p dx dy \\ & \quad + \sum_{|i|=1}^{2^K-1} \sum_{|j|=1}^{2^K-1} \int_{\pi i 2^{-K}}^{\pi(i+1)2^{-K}} \int_{\pi j 2^{-K}}^{\pi(j+1)2^{-K}} \sup_{n, m < r_{i,j}} |\sigma_{n,m}^{\alpha, \beta} a(x, y)|^p dx dy \\ & = (C) + (D) \end{aligned}$$

where $r_{i,j} := \left\lceil \frac{2^K}{|ij|^{\delta/(\alpha+\beta)}} \right\rceil$ with $\delta > 0$ chosen later. We suppose again that $i, j > 0$.

The term (C) was estimated in [15] for $\alpha = \beta = 1$. For arbitrary α and β we have by (5) and (14),

$$\begin{aligned} |\sigma_{n,m}^{\alpha,\beta} a(x, y)| &\leq C_p 2^{2K/p} \int_I \frac{1}{n^\alpha |x-t|^{\alpha+1}} dt \int_J \frac{1}{m^\beta |y-u|^{\beta+1}} du \\ &\leq C_p \frac{2^{2K/p+K\alpha+K\beta}}{n^\alpha m^\beta i^{\alpha+1} j^{\beta+1}} \end{aligned}$$

whenever $x \in [\pi i 2^{-K}, \pi(i+1)2^{-K})$ and $y \in [\pi j 2^{-K}, \pi(j+1)2^{-K})$. Therefore

$$\begin{aligned} (C) &\leq C_p \sum_{i=1}^{2^K-1} \sum_{j=1}^{2^K-1} 2^{-2K} \frac{2^{2K+K\alpha p+K\beta p}}{r_{ij}^{\alpha p+\beta p} i^{(\alpha+1)p} j^{(\beta+1)p}} \\ &\leq C_p \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{i^{(\alpha+1)p-\delta p} j^{(\beta+1)p-\delta p}} \end{aligned}$$

which converges if

$$(19) \quad \delta < \frac{(\alpha+1)p-1}{p} \quad \text{and} \quad \delta < \frac{(\beta+1)p-1}{p}.$$

Using (17) and integrating by parts in both variables we get that

$$\begin{aligned} &\int_I \int_J a(t, u) K_n^\alpha(x-t) K_m^\beta(y-u) dt du \\ &= - \int_J A(\mu, u) K_n^\alpha(x-\mu) (K_m^\beta)'(y-u) du \\ &\quad + \int_I A(t, \mu) (K_n^\alpha)'(x-t) K_m^\beta(y-\mu) dt \\ &\quad - \int_I \int_J A(t, u) (K_n^\alpha)'(x-t) (K_m^\beta)'(y-u) dt du \\ &=: D_{n,m}^1(x, y) + D_{n,m}^2(x, y) + D_{n,m}^3(x, y) \end{aligned}$$

because $A(\mu, -\mu) = A(\mu, \mu) = 0$.

Applying (6), (9), (16) and (14) we derive

$$\begin{aligned} & |D_{n,m}^1(x, y)| \\ & \leq C_p 2^{2K/p-2K} \frac{n^{\eta+\alpha(\eta-1)}}{|x-\mu|^{(\alpha+1)(1-\eta)}} 2^{-K} \frac{m^{\theta+1+\beta(\theta-1)}}{|y-u|^{(\beta+1)(1-\theta)}} \\ & \leq C_p 2^{2K/p-3K} n^{\eta+\alpha(\eta-1)} \left(\frac{2^K}{i}\right)^{(\alpha+1)(1-\eta)} m^{\theta+1+\beta(\theta-1)} \left(\frac{2^K}{j}\right)^{(\beta+1)(1-\theta)} \end{aligned}$$

provided that $x \in [\pi i 2^{-K}, \pi(i+1)2^{-K})$ and $y \in [\pi j 2^{-K}, \pi(j+1)2^{-K})$.

Choosing

$$\eta := \frac{2\alpha-1}{2(\alpha+1)} \quad \text{and} \quad \theta := \frac{2\beta-1}{2(\beta+1)}$$

we obtain

$$|D_{n,m}^1(x, y)| \leq C_p 2^{2K/p} \frac{1}{i^{3/2}} \frac{1}{j^{3/2}}.$$

Thus

$$\int_{\mathbf{T} \setminus 4I} \int_{\mathbf{T} \setminus 4J} \sup_{n,m < r_{i,j}} |D_{n,m}^1(x, y)|^p dx dy \leq C_p \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} 2^{-2K} 2^{2K} \frac{1}{i^{3p/2}} \frac{1}{j^{3p/2}}$$

and this is convergent if $p > 2/3$. Note that this is the best possible result which can be obtained in this way. The analogous estimation for $|D_{n,m}^2(x, y)|$ can be proved similarly.

To estimate $|D_{n,m}^3(x, y)|$ use (8), (14) and (16) and observe that

$$\begin{aligned} |D_{n,m}^3(x, y)| & \leq C_p 2^{2K/p-2K} \int_I \frac{1}{n^{\alpha-1}|x-t|^{\alpha+1}} dt \int_J \frac{1}{m^{\beta-1}|y-u|^{\beta+1}} du \\ & \leq C_p \frac{2^{2K/p-2K+K\alpha+K\beta} n^{1-\alpha} m^{1-\beta}}{i^{\alpha+1} j^{\beta+1}} \end{aligned}$$

for $x \in [\pi i 2^{-K}, \pi(i+1)2^{-K})$ and $y \in [\pi j 2^{-K}, \pi(j+1)2^{-K})$. So

$$\begin{aligned} & \int_{\mathbf{T} \setminus 4I} \int_{\mathbf{T} \setminus 4J} \sup_{n,m < r_{i,j}} |D_{n,m}^3(x, y)|^p dx dy \\ & \leq C_p \sum_{i=1}^{2^K-1} \sum_{j=1}^{2^K-1} 2^{-2K} \frac{2^{2K-2Kp+K\alpha p+K\beta p} r_{i,j}^{(2-\alpha-\beta)p}}{i^{(\alpha+1)p} j^{(\beta+1)p}} \\ & \leq C_p \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{i^{\frac{\delta(2-\alpha-\beta)p}{\alpha+\beta} + (\alpha+1)p}} \frac{1}{j^{\frac{\delta(2-\alpha-\beta)p}{\alpha+\beta} + (\beta+1)p}}. \end{aligned}$$

This series converges if

$$(20) \quad \delta > (\alpha + \beta) \frac{1 - (\alpha + 1)p}{(2 - \alpha - \beta)p} \quad \text{and} \quad \delta > (\alpha + \beta) \frac{1 - (\beta + 1)p}{(2 - \alpha - \beta)p}$$

if $\alpha + \beta \neq 2$. In case $\alpha = \beta = 1$ we have $p > 1/(\alpha + 1)$. It is easy to see that (19) and (20) imply $p > 1/(\alpha + 1)$ and $p > 1/(\beta + 1)$. In this case we have shown that

$$(21) \quad \int_{\mathbf{T} \setminus 4I} \int_{\mathbf{T} \setminus 4J} |\sigma_*^{\alpha, \beta} a(x, y)|^p dx dy \leq C_p.$$

Step 3. Integrating over $4I \times (\mathbf{T} \setminus 4J)$. This case is analogous to Step 1.

Combining (18) and (21) we proved that $\sigma_*^{\alpha, \beta}$ is p -quasi-local for each $p_0 < p \leq 1$. Theorems A and B complete the proof of (12). $\square$

Note that Theorem 1 was proved in [15] for greater p_0 and under some strong conditions on α .

We can state the same for the maximal conjugate (C, α) operator.

Theorem 2. *Assume that $i = 0, 1, \dots, d$ and $0 < \alpha_k \leq 1$ ($k = 1, \dots, d$). Then*

$$\|\tilde{\sigma}_*^{(i); \alpha} f\|_{p, q} \leq C_{p, q} \|f\|_{H_{p, q}} \quad (f \in H_{p, q})$$

for every $p_0 < p < \infty$ and $0 < q \leq \infty$. Especially, if $f \in L_1$ then

$$\lambda(\tilde{\sigma}_*^{(i); \alpha} f > \rho) \leq \frac{C}{\rho} \|f\|_1 \quad (\rho > 0).$$

PROOF. By Theorem 1 for $p = q$, (2) and (11) we obtain

$$\|\tilde{\sigma}_*^{(i); \alpha} f\|_p = \|\sigma_*^\alpha \tilde{f}^{(i)}\|_p \leq C_p \|\tilde{f}^{(i)}\|_{H_p} \leq C_p \|f\|_{H_p} \quad (f \in H_p)$$

for every $p_0 < p < \infty$. The first inequality of Theorem 2 follows from Theorem A.

Let us point out this inequality for $p = 1$ and $q = \infty$. If $f \in L_1$ then (1) implies

$$\|\tilde{\sigma}_*^{(i); \alpha} f\|_{1, \infty} = \sup_{\rho > 0} \rho \lambda(\sigma_*^{(i); \alpha} f > \rho) \leq C \|f\|_{H_{1, \infty}} \leq C \|f\|_1$$

which shows the weak type inequality in Theorem 2. The proof of the theorem is complete. $\square$

Since the trigonometric polynomials are dense in L_1 , the weak type inequality of Theorem 2 and the usual density argument (see MARCINKIEWICZ, ZYGMUND [7]) imply

Corollary 1. Assume that $i = 0, 1, \dots, d$ and $0 < \alpha_k \leq 1$ ($k = 1, \dots, d$). If $f \in L_1$ then

$$\tilde{\sigma}_n^{(i); \alpha} f \rightarrow \tilde{f}^{(i)} \quad \text{a.e.}$$

as $\min(n_1, \dots, n_d) \rightarrow \infty$ and $2^{-\tau} \leq n_k/n_j \leq 2^\tau$ ($k, j = 1, \dots, d$).

Note that Theorem 2 and Corollary 1 for $i = 0$ were proved under some conditions on α (see [15]). For other i 's these are new results since $\tilde{f}^{(i)}$ is not necessarily integrable whenever f is.

Now we consider the norm convergence of $\sigma_n^\alpha f$. It follows from (12) that $\sigma_n^\alpha f \rightarrow f$ in L_p norm as $n \rightarrow \infty$ if $f \in L_p$ ($1 < p < \infty$). We are going to generalize this result.

Theorem 3. Assume that $i = 0, 1, \dots, d$ and $0 < \alpha_k \leq 1$ ($k = 1, \dots, d$). If $n \in \mathbf{N}^d$ is in the cone, i.e. $2^{-\tau} \leq n_k/n_j \leq 2^\tau$ for all $k, j = 1, \dots, d$, then

$$\|\tilde{\sigma}_n^{(i); \alpha} f\|_{H_{p,q}} \leq C_{p,q} \|f\|_{H_{p,q}} \quad (f \in H_{p,q})$$

whenever $p_0 < p < \infty$ and $0 < q \leq \infty$.

PROOF. Since $(\sigma_n^\alpha f)^{\sim(i)} = \tilde{\sigma}_n^{(i); \alpha} f$, we have by Theorems 1 and 2 that

$$\|(\sigma_n^\alpha f)^{\sim(i)}\|_p \leq C_p \|f\|_{H_p} \quad (f \in H_p).$$

(3) implies that

$$\|\tilde{\sigma}_n^{(i); \alpha} f\|_{H_p} \leq C_p \|f\|_{H_p} \quad (f \in H_p).$$

Now Theorem A proves Theorem 3. $\square$

Corollary 2. Suppose that $i = 0, 1, \dots, d$, $0 < \alpha_k \leq 1$ ($k = 1, \dots, d$), $p_0 < p < \infty$ and $0 < q \leq \infty$. If $f \in H_{p,q}$ then

$$\tilde{\sigma}_n^{(i); \alpha} f \rightarrow \tilde{f} \quad \text{in } H_{p,q} \text{ norm}$$

as $\min(n_1, \dots, n_d) \rightarrow \infty$ and $2^{-\tau} \leq n_k/n_j \leq 2^\tau$ ($k, j = 1, \dots, d$).

We suspect that Theorems 1, 2 and 3 for $p \leq p_0$ are not true though we could not find any counterexample.

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