

Generalized pseudo-contractions and nonlinear variational inequalities

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Abstract. Based on a modified iterative algorithm, the solvability of a class of nonlinear variational inequality problems involving Lipschitzian generalized pseudo-contractions is presented on convex sets in Hilbert spaces.

1. Introduction

General variational inequalities have been applied to many problems in applied mathematics, physics, engineering sciences, and others. A closely associated notion of the complementarity involves several problems in mathematical programming, game theory, economics, and mechanics. There are situations where both concepts are equivalent, especially on a closed convex cone. For more details on variational inequalities, we advise to consult [2–4, 7–12].

Let H be a real Hilbert space and let K be a nonempty closed convex subset of H . Let $\langle u, v \rangle$ and $\|u\|$ denote, respectively, the inner product and norm on H for u, v in H . Let P_K be the projection of H onto K . For an operator $T : K \rightarrow H$, we consider the nonlinear variational inequality (NVI) problem (Pl): Find an element x in K such that

$$(1) \quad \langle (I - T)x, y - x \rangle \geq 0 \quad \text{for all } y \text{ in } K,$$

where I is the identity.

Mathematics Subject Classification: 49J40, 90C20.

Key words and phrases: generalized pseudocontraction, iterative algorithm, variational inequality problem, complementarity problem.

The NVI problem (1) is equivalent to a complementarity problem when K is a closed convex cone ([9]).

Next, we consider an important concept of the generalized pseudo-contractivity – a mild generalization of the pseudo-contractivity introduced by BROWDER and PETRYSHYN in [1]. Generalized pseudo-contractions are more general than Lipschitz continuous operators and unify certain class of operators.

Definition 1.1. An operator $T : H \rightarrow H$ is said to be a generalized pseudo-contraction if, for all x, y in H , there exists a constant $r > 0$ such that

$$(2) \quad \|Tx - Ty\|^2 \leq r^2 \|x - y\|^2 + \|Tx - Ty - r(x - y)\|^2.$$

It is easy to check that (2) is mutually equivalent to

$$(3) \quad \langle Tx - Ty, x - y \rangle \leq r \|x - y\|^2.$$

Clearly, this implies that

$$(4) \quad \langle (I - T)x - (I - T)y, x - y \rangle \geq (1 - r) \|x - y\|^2,$$

that is, $I - T$ is strongly monotone for $r < 1$. Here I is the identity.

For $r = 1$ in (2), we arrive at the usual concept of the pseudo-contractivity of T introduced by BROWDER and PETRYSHYN in [1], that is,

$$(5) \quad \|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - Ty - (x - y)\|^2.$$

An operator $T : H \rightarrow H$ is called Lipschitz continuous if there is a constant $s > 0$ such that

$$(6) \quad \|Tx - Ty\| \leq s \|x - y\| \quad \text{for all } x, y \text{ in } H.$$

Clearly, (6) implies

$$(7) \quad \langle Tx - Ty, x - y \rangle \leq s \|x - y\|^2.$$

Remark 1.1. We note that (2) and (3) are mutually equivalent, whereas (6) and (7) are not (since (7) does not imply (6)). That is why the generalized pseudo-contractions are more general than the Lipschitz continuous operators.

Here our aim is to present, based on a modified iterative algorithm, the solution of the NVI problem (1) involving the generalized pseudo-contractions which are Lipschitz continuous. The obtained results generalize, especially the results on pseudo-contractive and Lipschitz continuous operators in Hilbert space settings. For selected recent research works on the pseudo-contractivity, we advise [5, 6].

2. Nonlinear variational inequalities

We are just about ready to present the result on the solvability of the NVI problem (1).

Lemma 2.1 [4]. *Let K be a nonempty closed convex subset of a real Hilbert space H . Then for an element z in H , an element x in K satisfies*

$$(8) \quad \langle x - z, y - x \rangle \geq 0 \quad \text{for all } y \text{ in } K \quad \text{iff} \quad x = P_K z.$$

Theorem 2.1. *Let K be a nonempty closed convex subset of a Hilbert space H . Then NVI problem (1) has a solution x in K iff x in K satisfies the relation*

$$(9) \quad x = P_K[x - t(x - Tx)],$$

where $t > 0$ is arbitrary.

PROOF. Assume an element u in K is a solution of the NVI problem (1). Then u in K is such that

$$(10) \quad \langle u - Tu, y - u \rangle \geq 0 \quad \text{for all } y \text{ in } K.$$

Now for any $t > 0$, it follows that

$$(11) \quad \langle u - (u - t(u - Tu)), y - u \rangle \geq 0 \quad \text{for all } y \text{ in } K.$$

By Lemma 2.1, we find that

$$(12) \quad u = P_K[u - t(u - Tu)].$$

Conversely, if u satisfies the relation

$$u = P_K[u - t(u - Tu)],$$

then u belongs to K and, by Lemma 2.1, we obtain

$$\langle u - (u - t(u - Tu)), y - u \rangle \geq 0 \quad \text{for all } y \text{ in } K.$$

Since $t > 0$, this implies that

$$\langle u - Tu, y - u \rangle \geq 0 \quad \text{for all } y \text{ in } K.$$

Hence u is a solution of the NVI problem (1).

Theorem 2.2. *Let H be a real Hilbert space and K be a nonempty closed convex subset of H . Let $T : K \rightarrow H$ be generalized pseudo-contractive (with constant $r > 0$) and Lipschitz continuous (with constant $s \geq 1$). Let $\{a_n\}$ be an increasing sequence in $[0, 1)$ such that*

$$(13) \quad \sum_{n=0}^{\infty} a_n = \infty \quad \text{for all } n \geq 0.$$

If, for an element x_0 in K , the sequence $\{x_n\}$ is generated by an iterative algorithm

$$(14) \quad x_{n+1} = (1 - a_n)x_n + a_n P_K[(1 - t)x_n + tTx_n] \quad \text{for all } n \geq 0,$$

then the sequence $\{x_n\}$ converges to a unique solution of the NVI problem (1) for $0 < t < 2(1 - r)/(1 - 2r + s^2)$, and $r < 1$.

For $\{a_n\} = 1$, Theorem 2.2 reduces to

Corollary 2.1. *Let $T : K \rightarrow H$ be generalized pseudo-contractive and Lipschitz continuous, and let $r > 0$ and $s > 1$ be constants of the generalized pseudo-contractivity and Lipschitz continuity of T , respectively. Then the sequence $\{x_n\}$, generated by an iterative algorithm*

$$(15) \quad x_{n+1} = P_K[(1 - t)x_n + tTx_n] \quad \text{for an element } x_0 \text{ in } K$$

and for all t such that $0 < t < 2(1 - r)/(1 - 2r + s^2)$, converges to a unique solution of the NVI problem (1).

PROOF of Theorem 2.2. Suppose that z is a solution of the NVI problem (1). Then by Theorem 2.1, we have

$$z = P_K[(1 - t)z + tTz].$$

Since P_K is nonexpansive, we find that

$$(16) \quad \begin{aligned} \|x_{n+1} - z\| &= \|(1 - a_n)x_n + a_n P_K [(1 - t)x_n + tTx_n] - z\| \\ &\leq (1 - a_n)\|x_n - z\| + a_n \|t(Tx_n - Tz) + (1 - t)(x_n - z)\|. \end{aligned}$$

Now, since T is generalized pseudo-contractive (and hence equivalent to (3)) and Lipschitz continuous, it follows that

$$(17) \quad \begin{aligned} &\|t(Tx_n - Tz) + (1 - t)(x_n - z)\|^2 \\ &= (1 - t)^2\|x_n - z\|^2 + 2t(1 - t)\langle Tx_n - Tz, x_n - z \rangle + t^2\|Tx_n - Tz\|^2 \\ &\leq (1 - t)^2\|x_n - z\|^2 + 2t(1 - t)r\|x_n - z\|^2 + t^2s^2\|x_n - z\|^2 \\ &= [(1 - t)^2 + 2t(1 - t)r + t^2s^2] \|x_n - z\|^2. \end{aligned}$$

Applying (17) to (16), we get

$$(18) \quad \begin{aligned} \|x_{n+1} - z\| &\leq \left[1 - a_n + a_n \left((1 - t)^2 + 2t(1 - t)r + t^2s^2\right)^{1/2}\right] \|x_n - z\| \\ &= [1 - (1 - k)a_n] \|x_n - z\| \leq \prod_{j=0}^n [1 - (1 - k)a_j] \|x_0 - z\|, \end{aligned}$$

where $0 < k = [(1 - t)^2 + 2t(1 - t)r + t^2s^2]^{1/2} < 1$ for all t such that $0 < t < 2(1 - r)/((1 - 2r + s^2))$, $r < 1$ and $s \geq 1$. Since $\sum_{j=0}^{\infty} a_j = \infty$ and

$k < 1$, this implies that $\lim_{n \rightarrow \infty} \prod_{j=0}^n [1 - (1 - k)a_j] = 0$. Hence $\{x_n\}$ converges to z .

To show the uniqueness of the solution, let x_1 and x_2 be two solutions of the NVI problem (1). Then we have

$$(19) \quad \langle (I - T)x_1, y - x_1 \rangle \geq 0 \quad \text{for all } y \text{ in } K,$$

and

$$(20) \quad \langle (I - T)x_2, y - x_2 \rangle \geq 0 \quad \text{for all } y \text{ in } K.$$

If we replace y in (19) by x_2 and y in (20) by x_1 , and add, we obtain

$$(21) \quad \langle (I - T)x_1 - (I - T)x_2, x_1 - x_2 \rangle \leq 0.$$

Since $I - T$ is strongly monotone with constant $1 - r$, we find on applying (21) that

$$(22) \quad (1 - r)\|x_1 - x_2\|^2 \leq \langle (I - T)x_1 - (I - T)x_2, x_1 - x_2 \rangle \leq 0.$$

This implies that $x_1 = x_2$, and this completes the proof.

Acknowledgement. The author wishes to express his sincere appreciation to the referee for all the valuable comments leading to the revised version.

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(Received May 14, 1996; revised February 28, 1997)