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On arrangment of regular cyclic subgroup in symmetric group

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Abstract. In this paper we investigate weakly normal and polinormal regular cyclic subgroups in a symmetric group S_n. We give here necessary and sufficient conditions for the subgroup generated by a long cycle $(\overline{0}\ \overline{1} \dots \overline{n-1})$ in S_n to be weakly normal or polinormal. We also describe a normalizer of this subgroup in S_n.

Let $Z_n = \{\overline{0}\ \overline{1} \dots \overline{n-1}\}$ be a ring of all integers modulo n (where n is a fixed positive integer) and let S_n be a symmetric group of degree n which acts on Z_n. As usual, Z_n^* denotes a set of invertible elements in Z_n. Each cycle of the length n in S_n is called a long cycle. The subgroup generated by a long cycle is called regular. We denote $x^y = y^{-1}xy$.

Let G be an arbitrary group, and D be its subgroup. We denote by $N_G(D)$ a normalizer of D in G. If $A, B \subseteq G$ then $\langle A, B\rangle$ denotes the subgroup generated by A, B. If $g \in G$ then $D^{\langle g,D\rangle}$ denotes the normal closure of D in the group $\langle g, D\rangle$, that is $D^{\langle g,D\rangle}$ is the subgroup generated by all elements d^h, where $d \in D$ and $h \in \langle g, D\rangle$. The subgroup $D^{D^{\langle g,D\rangle}}$ denotes a normal closure of D in $D^{\langle g,D\rangle}$ and we have $D^{D^{\langle g,D\rangle}} \subseteq D^{\langle g,D\rangle}$.

A subgroup D of an arbitrary group G is called *weakly normal* [3] if $D^g \leq N_G(D)$ implies $D^g = D$. In paper [1] Z. I. Borevich and O. Macedońska introduced definition of a polinormal subgroup. The subgroup D is *polinormal* in G if for each $g \in G$ the following property holds: $D^{\langle g,D\rangle} = D^{D^{\langle g,D\rangle}}$. Finally, a subgroup D is called *pronormal* [3]

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if subgroups D and D^g are conjugated in the subgroup $\langle D, D^g \rangle$, for each $g \in G$. Sylow subgroups in a finite group are pronormal. Every pronormal subgroup is weakly normal and polinormal.

In this paper we investigate weakly normal and polinormal regular cyclic subgroups in a symmetric group S_n. We give here the necessary and sufficient conditions for the subgroup generated by a long cycle $(\overline{0}\,\overline{1}\ldots\overline{n-1})$ in S_n to be weakly normal and polinormal. We also describe a normalizer of this subgroup in S_n.

If n is a prime then the regular subgroup generated by a long cycle is a Sylow subgroup in S_n, so it is pronormal. R. I. Tyshkevich proved [5] that if $(n, \varphi(n)) = 1$ (where φ is the Euler function), then the subgroup generated by a long cycle in S_n is pronormal. The same result, by another methods, was obtained by P. P. Palfy in [4].

Let $\gamma = (\overline{0}\,\overline{1}\ldots\overline{n-1})$ be a long cycle in S_n and let Γ be the subgroup generated by γ. A permutation η from Γ is a long cycle (i.e. it is conjugated to γ), if $\eta = \gamma^k$, where k is an invertible element of Z_n (that is, $k \in Z_n^*$). We describe a normalizer N of Γ in S_n.

Lemma 1. *The normalizer N of Γ in S_n consists of all such permutations η, which act on Z_n as follows:*

$$\eta(x) = kx + a,$$

where $k \in Z_n^*$, $a \in Z_n$. Moreover:

$$|N| = n \cdot \varphi(n).$$

Proof. The permutation γ acts on Z_n as follows:

$$\gamma(x) = x + \overline{1}.$$

Hence, we have: $\gamma^k(x) = x + k$. A permutation η belongs to N if and only if:

$$\eta\gamma\eta^{-1} = \gamma^k,$$

for some $k \in Z_n^*$.

Let η be in N, then we have:

$$\eta(x + \overline{1}) = \eta(\gamma(x)) = \eta\gamma\eta^{-1}\eta(x) = \gamma^k\eta(x) = \eta(x) + k,$$

thus by induction

$$\eta(x+y) = \eta(x) + ky.$$

If

$$\eta(\overline{0}) = a, \text{ then } \eta(y) = ky + a, \text{ and } \eta^{-1}(y) = k^{-1}y - k^{-1}a.$$

Conversely, let $\eta(x)$ be equal to $kx + a$, where $k \in Z_n^*$, $a \in Z_n$. Hence

$$\begin{aligned}\eta\gamma\eta^{-1}(x) &= \eta\gamma(k^{-1}x - k^{-1}a + \overline{1}) \\ &= \eta(k^{-1}x - k^{-1}a + \overline{1}) = x + k = \gamma^k(x),\end{aligned}$$

which finishes the proof. □

A weak normality of Γ in S_n is equivalent to the following condition: if η is a long cycle that belongs to N, then η belongs to Γ. If n is a prime, then Γ has $n-1$ long cycles and there are no long cycles in $N \setminus \Gamma$.

If $n = p_1^{\alpha_1} \dots p_m^{\alpha_m}$ is a factorization of n, then the ring Z_n is isomorphic to a direct product of rings Z_{q_i}, where $q_i = p_i^{\alpha_i}$. Let S_{q_i} be a symmetric group acting on Z_{q_i}, let Γ_i be a subgroup generated in S_{q_i} by a cycle $(\overline{0}\,\overline{1}\dots\overline{q_i - 1})$ and let N_i be the normalizer of Γ_i in S_{q_i}.

Lemma 2. *The normalizer N of a subgroup Γ in S_n is isomorphic to the direct product $N_1 \times \dots \times N_m$.*

Proof. Let ψ be a mapping:

$$\psi : N \to N_1 \times \cdots \times N_m : \psi(\eta) = (\eta_1, \dots, \eta_m),$$

such, that if $\eta(x) = kx + a$ for $k \in Z_n^*$, $a \in Z_n$, then for all $i \in \{1, \dots, m\}$ $\eta_i(x) = kx + a$, where all numbers are taken modulo q_i. Clearly, ψ is a homomorphism. Let η, $\mu \in N$ and

$$\eta(x) = kx + a, \quad \mu(x) = lx + b.$$

If $\psi(\eta) = \psi(\mu)$ then $kx + a \equiv lx + b \pmod{q_i}$ for $i = 1 \dots m$, $x \in Z$. Hence $k \equiv l$, $a \equiv b \pmod{q_i}$, so $k = l$, $a = b$ in Z_n and this means that ψ is a monomorphism. We know from Lemma 1 that $|N| = n \cdot \varphi(n)$. Let us compute:

$$|N_1 \times \dots \times N_m| = q_1 \dots q_m \cdot \varphi(q_1) \dots \varphi(q_m) = n \cdot \varphi(n),$$

where the last equation holds from multiplicative property of the Euler function. It means that φ is a monomorphism of two groups, which have the same number of elements, so φ is an isomorphism. □

In the notations of Lemma 2 we get:

Lemma 3. *A permutation $\eta \in N$ is a long cycle, if and only if, each η_i is a long cycle in N_i. A number of long cycles in N is a product of numbers of long cycles in all N_i.*

PROOF. If η is a long cycle, which belongs to N, then n is equal to the least common multiple of orders of all permutations η_i. Lengths of different η_i are coprime, so each η_i must be a long cycle in N_i. The converse statement is clear. Thus in N we have as many long cycles as rows of the type $(\eta_1, \dots, \eta_m)$, where η_i is a long cycle in N_i. □

The following theorem was firstly presented without proof in [2].

Theorem 1. *A subgroup Γ generated by a long cycle in S_n is weakly normal if and only if n is not divisible by 8 and is not divisible by a square of an odd prime.*

POOF. We note, that if $n = 4$, then there are only two long cycles in N: γ and γ^3, and these cycles belong to Γ. Let n be not divisible by 8 and by a square of an odd prime. Then for an arbitrary diviser p_i of n, there exist $\varphi(q_i)$ long cycles in a subgroup N_i. Hence there are $\varphi(q_1) \dots \varphi(q_m) = \varphi(n)$ long cycles in N and each of them belongs to Γ. So we proved that Γ is weakly normal in S_n.

Let now n be divisible by p^2 or by 8, where p is an odd prime. We want to show that Γ is not weakly normal. For this purpose we use the permutation acting on Z_n:

$$\sigma(x) = x + \frac{x(x + \overline{1})}{\overline{2}} \frac{n}{k},$$

where $k = p$ if n is divisible by p^2 and $k = 2$ if n is divisible by 8. Then the inverse permutation is:

$$\sigma^{-1}(x) = x - \frac{x(x + \overline{1})}{\overline{2}} \frac{n}{k}.$$

The permutation $\eta = \sigma\gamma\sigma^{-1}$ is a long cycle and

$$\eta(x) = \left(\overline{1} + \frac{n}{k}\right) x + \overline{1} + \frac{n}{k}.$$

By Lemma 1, permutation belongs to N and does not belong to Γ. So Γ is not a weakly normal subgroup of S_n, which finishes the proof. □

Corollary 1. *If n is divisible by 8 or by a square of an odd prime, then Γ is not a polinormal subgroup of S_n.*

Proof. Let σ and η be the same permutations as in the proof of Theorem 1. We show that a permutation σ normalizes the subgroup $\langle\gamma,\eta\rangle$. Indeed by computation we get $\sigma\eta\sigma^{-1} = \gamma^{(-1+\frac{n}{p})}\eta^2 \in \langle\gamma,\eta\rangle$ and by definition of η we have $\sigma\gamma\sigma^{-1} = \eta \in \langle\gamma,\eta\rangle$. Hence $\langle\gamma,\eta\rangle \subseteq \Gamma^{\langle\sigma,\Gamma\rangle}$. So we get $\Gamma^{\langle\sigma,\Gamma\rangle} = \langle\gamma^h; h \in \langle\sigma\rangle\rangle = \langle\gamma,\eta\rangle$. The subgroup $\Gamma^{\Gamma^{\langle\sigma,\Gamma\rangle}}$ is equal to $\Gamma^{\langle\gamma,\eta\rangle}$ and by Theorem 1 $\langle\gamma,\eta\rangle \subseteq N$, so we have $\Gamma^{\langle\gamma,\eta\rangle} = \Gamma$, which means that Γ is not polinormal in S_n. $\square$

The above theorem implies straightforward, such a corollary:

Corollary 2. *If Γ is polinormal in S_n then it is also weakly normal.*

By simple calculation it follows that for $n = 6$ the subgroup Γ is not polinormal in S_n. So the converse to Corollary 2 is not true. We note that $(6, \varphi(6)) = 2 > 1$, and we can prove:

Theorem 2. *A subgroup Γ is polinormal in S_n if and only if $(n, \varphi(n)) = 1$ or $n = 4$.*

Before we prove the above Theorem, we will give some auxiliary lemmas. Let, for now, n be an even integer grater than 4. So there exists k, such that $n = 2k$. We take $\gamma = (\overline{0}\ \overline{1} \ldots \overline{2k-1})$ and $g = (\overline{3}\ \overline{2k-1})(\overline{5}\ \overline{2k-3}) \ldots$. The permutation g is an involution. We also introduce two symbols x and y for following cycles:

$$x = (\overline{0}\,\overline{2} \ldots \overline{2k-2}),\ y = (\overline{1}\,\overline{3} \ldots \overline{2k-1}).$$

Permutations x and y commute and have the same order k. We, also, have a relation $\gamma^2 = xy$. Let Γ be the cyclic subgroup generated by γ. The normal closure of $\Gamma = \langle\gamma\rangle$ in the group $\langle\gamma, g\rangle$ is generated by all conjugates of γ, so $\langle\gamma\rangle^{\langle\gamma,g\rangle} = \langle\gamma^h;\ h \in \langle\gamma, g\rangle\rangle$. It is enough to take as generators of this normal closure all elements γ^h for $h \in \langle g\rangle$. The element g is an involution, so:

$$\Gamma^{\langle g,\Gamma\rangle} = \langle\gamma\rangle^{\langle g,\gamma\rangle} = \langle\gamma^h \mid h \in \langle g\rangle\rangle = \langle\gamma, \gamma^g\rangle.$$

Now we describe the subgroup $\Gamma^{\Gamma^{\langle g,\Gamma\rangle}}$ which, by the above is:

$$\Gamma^{\Gamma^{\langle g,\Gamma\rangle}} = \langle\gamma\rangle^{\langle\gamma,\gamma^g\rangle}, \tag{1}$$

and we prove, it is generated by γ and γ^{γ^g}.

Lemma 4. *The permutations g and γ satisfy the relation:*

$$(\gamma^g)^2 = \gamma \cdot \gamma^{\gamma^g} \tag{2}$$

and for all numbers $l \in Z$ we have:

$$\gamma^{(\gamma^g)^l} \in \langle \gamma, \gamma^{\gamma^g} \rangle. \tag{3}$$

Proof. By calculation we get $(\gamma)^2 = xy^{-1} = \gamma \cdot \gamma^{\gamma^g}$ and it proves the first part of the lemma. We prove the second part by induction on l. Let us start from $l = 2$. By equation (2):

$$\gamma^{(\gamma^g)^2} = \gamma^{\gamma \cdot \gamma^{\gamma^g}} \in \langle \gamma, \gamma^{\gamma^g} \rangle.$$

Let, now (3) be true for all numbers less than l, and let $l \geq 2$ then $\gamma^{(\gamma^g)^l} = (\gamma^{(\gamma^g)^{l-2}})^{(\gamma^g)^2}$, by inductive assumption $(\gamma^{(\gamma^g)^{l-2}}) \in \langle \gamma, \gamma^{\gamma^g} \rangle$ and by the equation (2) $(\gamma^g)^2 \in \langle \gamma, \gamma^{\gamma^g} \rangle$, which finishes the proof. □

We have to show that Γ is not polinormal. Due to the second part of the Lemma 4 and from the equation (1) we get:

$$\Gamma^{\Gamma^{\langle g, \Gamma \rangle}} = \langle \gamma, \gamma^{\gamma^g} \rangle.$$

We need to show, that:

$$\Gamma^{\langle g, \Gamma \rangle} \not\subseteq \Gamma^{\Gamma^{\langle g, \Gamma \rangle}}.$$

In fact, it is enough to show that:

$$\gamma^g \notin \Gamma^{\Gamma^{\langle , \Gamma \rangle}} = \langle \gamma, \gamma^{\gamma^g} \rangle. \tag{4}$$

Let us denote $\delta = \gamma^{\gamma^g} = (\overline{0}\ \overline{1}\ \overline{2k-2}\ \overline{2k-1} \ldots \overline{5}\ \overline{2}\ \overline{3})$. By computations one can find relations between γ and δ:

$$\gamma^{2k} = \delta^{2k} = 1,$$

$$(\gamma \cdot \delta)^k = (xy^{-1})^k = x^k y^{-k} = 1,$$

$$\gamma^2 \cdot \delta^2 = xy \cdot x^{-1} y^{-1} = 1.$$

Now we want to describe groups with such relations:

Lemma 5. *Let G be a group with presentation:*

$$\langle a, b \mid a^{2k} = b^{2k} = (ab)^k = 1,\ a^2b^2 = 1\rangle \tag{5}$$

Then:

(1) every element of G can be written in the form $a^s c^t$, where: $c = ab$, $s \in \{1, \dots, 2k\}$, $t \in \{1, \dots, k\}$,

(2) $|G| \leq 2k^2$,

(3) if G has such an element d that $a^d = b$ then G has a relation $a^2 = c^s$, where s is an odd integer.

Proof. First, we want to describe the commutator subgroup G' in G. The group G is two-generator, the commutator subgroup G' is generated by elements $[a^s, b^t]$, $s, t \in Z$. From the relation $a^2b^2 = 1$, it follows the relation $a^2b = ba^2$. Indeed it follows from equalities: $a^2b = b^{-2}b = bb^{-2} = ba^2$. So the commutator subgroup is cyclic and generated by $[a, b]$. From the relation $a^2b^2 = 1$ we have the equality $ab = a^{-1}b^{-1}$. Hence

$$[a, b] = a^{-1}b^{-1}ab = (ab)^2, \tag{6}$$

which means that $G' = \langle [a, b]\rangle \subseteq \langle ab\rangle$ and the subgroup $H = \langle ab\rangle$ is normal and has order k. The quotient goup G/H is cyclic generated by the coset aH and has order less or equal $2k$. So every element of G can be written as $a^s(ab)^t$ and G has the order less or equal $2k^2$. It proves (1) and (2).

To prove (3) we show, first, that in G holds the relation $ca = ac^{-1}$ (where $c = ab$). We use relations of G and the equality (6), to get:

$$ca = aba = a^2b[b, a] = aab(ab)^{-2} = a(ab)^{-1} = ac^{-1}. \tag{7}$$

Now we prove that for every l the following holds:

$$c^l a = ac^{-l}. \tag{8}$$

Let the equation be true for all numbers less or equal to l, then by inductive assumption and (7): $c^{l+1}a = cac^{-l} = ac^{-1}c^{-l} = c^{-(l+1)}$. Let now there exists $d \in G$, such that $a^d = b$. Then $d = a^s c^t$ and by (8) we have: $a^d = c^{-t}a^{-s}aa^s c^t = c^{-t}ac^t = ac^{2t} = b$. From the last equation we obtain the equality $a^2c^{2t} = ab = c$ and we get the relation $a^2 = c^s$ for s an odd integer. It finishes the third part of Lemma 5. □

Due to the previous Lemma we can prove that:

Corollary 3. *The element γ^g is not in $\langle \gamma, \gamma^{\gamma^g} \rangle$.*

Proof. The subgroup $\langle \gamma, \gamma^{\gamma^g} \rangle = \langle \gamma, \delta \rangle$ satisfies the same relations as the group G in Lemma 5. So by that Lemma γ^g belongs to $\langle \gamma, \gamma^{\gamma^g} \rangle$ if there is a relation: $\gamma^2 = \delta^{2s+1}$. We show that elements γ and δ do not satisfy such a relation. Indeed, $\gamma^2 = xy$, and $\delta^{2s+1} = (xy^{-1})^s \delta$. Suppose the relation $\gamma^2 = \delta^{2s+1}$ holds then we have: $\delta = x^{1-s} y^{s-1}$, which never holds because δ is a long cycle, while $x^{1-s} y^{s-1}$ is not a long cycle. $\square$

The following Corollary follows straightforward from Corollary 2 and from (4):

Corollary 4. *If n is an even number then the subgroup $\Gamma = \langle (\overline{0}\,\overline{1} \ldots \overline{n-1}) \rangle$ is not polinormal in S_n.*

Now we can prove the main Theorem 2.

Proof of Theorem 2. Let $n > 4$. If $(n, \varphi(n)) = 1$ then by Palfy–Tyshkievich Theorem ([4], [5]) the subgroup Γ is pronormal, so it is also polinormal. Conversely if Γ is polinormal then by Corollary 2 it is weakly normal. Hence n is not divisible by 8 and by a square of an odd prime number. It means that it is enough to prove that n cannot be an even number, which follows from Corollary 4. $\square$

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