

A note on multiplicative functions with regularity properties

By BUI MINH PHONG (Budapest)

J.L. MAUCLAIRE and L. MURATA [8] have shown that a multiplicative function $g(n)$ with properties

$$(1) \quad |g(n)| = 1 \quad (n = 1, 2, \dots)$$

and

$$(2) \quad \sum_{n \leq x} |g(n+1) - g(n)| = o(x) \quad \text{as } x \rightarrow \infty$$

has to be completely multiplicative. It is obvious that (1) and (2) hold for functions of the type

$$g(n) = n^{i\tau},$$

where τ is a real number. I. KÁTAI [6] conjectured that $g(n) = n^{i\tau}$ are the only multiplicative functions that satisfy the conditions (1) and (2). This conjecture remains open, some partial results are known. For such results we refer to A. HILDEBRAND [4], [5] and I. KÁTAI [7].

Our purpose in this note is to prove the following

Theorem. *Let A, B be positive integers and let C be a non-zero complex number. Assume that a complex-valued completely multiplicative function $g(n)$ satisfies the conditions*

$$(3) \quad |g(n)| = 1 \quad (n = 1, 2, \dots)$$

and

$$(4) \quad \sum_{n \leq x} |g(An + B) - Cg(n)| = o(x) \quad \text{as } x \rightarrow \infty.$$

If there is a positive integer k for which

$$(5) \quad \limsup_{x \rightarrow \infty} x^{-1} \left| \sum_{n \leq x} (g(n))^k \right| > 0,$$

then there are a real constant τ and a completely multiplicative function $G(n)$ such that

$$(6) \quad g(n) = n^{i\tau} \cdot G(n),$$

and

$$(7) \quad [G(n)]^k = 1$$

hold for all positive integers n , moreover

$$(8) \quad \sum_{n \leq x} |G(n+1) - G(n)| = o(x) \quad \text{as } x \rightarrow \infty.$$

Remarks. (i) In the special case when $A = B = C = k = 1$, our theorem can be deduced directly from Theorem 2 of A. HILDEBRAND [3]. In this case, by using HALÁSZ' theorem, it follows by (5) that for some real number τ

$$(9) \quad \operatorname{Re} \sum_P \frac{1 - g(p)p^{-i\tau}}{p} < \infty,$$

the series being taken over all primes p . A. HILDEBRAND [3] proved that (9) implies

$$(10) \quad \frac{1}{x} \sum_{n \leq x} \frac{g(n)}{g(n+1)} \rightarrow \prod_p F_p,$$

where

$$\mathbf{F}_p = 1 - \frac{2}{p} + 2 \left(1 - \frac{1}{p} \right) \operatorname{Re} \frac{g(p)p^{-i\tau}}{p - g(p)p^{-i\tau}}.$$

Thus, (3), (4), and (10) jointly imply that $\mathbf{F}_p = 1$ holds for each prime p , i.e.

$$g(p) = p^{i\tau}.$$

This shows that (6) holds with $G(n) \equiv 1$.

(ii) We hope that the conditions (3) and (4) imply (5), but we are unable to prove it presently. If we write a multiplicative function g satisfying (3) in the form $g = e^{2i\pi f}$, where f is an additive function, then it

is known from Chapter 8 of [1] that there are two possibilities: Either (5) holds for some positive integer k or $f(n)$ is uniformly distributed (mod 1).

We shall use some lemmas in the proof of our theorem.

For a given multiplicative function $g(n)$ we denote by $\mathbf{J} = \mathbf{J}(g)$ the set of those pairs (Q, R) of positive integers for which

$$(11) \quad \sum_{n \leq x} |g(Qn + R) - g(Qn)| = o(x) \quad \text{as } x \rightarrow \infty.$$

Lemma 1. *Assume that a completely multiplicative function $g(n)$ satisfies the conditions (3) and (4). Then $(Q, R) \in \mathbf{J}(g)$ for all fixed integers Q and R which satisfy the condition*

$$(12) \quad 0 < R < Q.$$

PROOF. We shall prove this lemma by the same method that was used in the proof of Lemma 2 in [10].

Assume that a completely multiplicative function $g(n)$ satisfies the conditions (3) and (4). Then, by using Theorem 1 of [9] and the complete multiplicativity of g , we have

$$(13) \quad g(A) = C.$$

Thus, $(A, B) \in \mathbf{J} = \mathbf{J}(g)$, and so $(A, 1) \in \mathbf{J}$.

We prove next the following assertions:

- (a) $(Q, 1) \in \mathbf{J}$ if $(q, 1) \in \mathbf{J}$ and $Q \geq q$
- (b) $(Q, R) \in \mathbf{J}$ if $(q, 1) \in \mathbf{J}$ and $0 < R < Q/(q - 1)$
- (c) $(h, 1) \in \mathbf{J}$ if $(h + 1, 1) \in \mathbf{J}$ and $h \geq 2$.

Assume that $(q, 1) \in \mathbf{J}$. By using the complete multiplicativity of g , we have

$$\begin{aligned} g[(q + 1)n + 1] - g[(q + 1)n] &= \frac{g(q + 1)}{g(q)} \{g(qn + 1) - g(qn)\} - \\ &\quad - \frac{1}{g(q)} \{g[q((q + 1)n + 1) + 1] - g[q((q + 1)n + 1)]\}, \end{aligned}$$

and so, by using (3) and the fact $(q, 1) \in \mathbf{J}$, we deduce that $(q + 1, 1) \in \mathbf{J}$. By using induction on q we have proved that (a) holds.

Assume again that $(q, 1) \in \mathbf{J}$. We shall prove (b) by using induction on R . From (a) it follows that (b) is satisfied for $R = 1$. Assume that $(Q, R) \in \mathbf{J}$ holds for all integers Q and R satisfying $0 < R < Q/(q - 1)$ and $R < R_0$. Let Q_0 be an integer such that

$$(14) \quad 0 < R_0 < Q_0/(q - 1).$$

In order to show (b) it suffices to prove that $(Q_0, R_0) \in \mathbf{J}$. Without loss of generality we may assume that $(Q_0, R_0) = 1$.

Let Q and R be positive integers such that

$$(15) \quad R_0 Q = Q_0 R + 1 \quad \text{and} \quad R < R_0.$$

It follows from (14) and (15) that

$$0 < R < (Q_0 R + 1)/Q_0 = R_0 Q/Q_0 < Q/(q-1).$$

Thus, by using our assumption and the fact $R < R_0$, we have $(Q, R) \in \mathbf{J}$. On the other hand, by (15), we get

$$\begin{aligned} g(Q_0 n + R_0) - g(Q_0 n) &= \frac{1}{g(Q)} [g(Q_0 Q n + R_0 Q) - g(Q)g(Q_0 n)] = \\ &= \frac{g(Q_0)}{g(Q)} \{g(Qn + R) - g(Qn)\} + \\ &\quad + \frac{1}{g(Q)} \{g[Q_0(Qn + R) + 1] - g[Q_0(Qn + R)]\}, \end{aligned}$$

consequently $(Q_0, R_0) \in \mathbf{J}$, because $(Q, R) \in \mathbf{J}$ and $(Q_0, 1) \in \mathbf{J}$. Thus, we have proved (b).

Finally, we prove (c). Assume that $(h+1, 1) \in \mathbf{J}$ and $h \geq 2$. Let

$$T(x) := \sum_{n \leq x} |g(hn + 1) - g(hn)|.$$

For each positive integer d with $0 \leq d \leq h-1$, we can choose positive integers $Q = Q(d)$ and $R = R(d)$ such that

$$(16) \quad (hd + 1)Q = h^2 R + 1.$$

We have

$$\begin{aligned} T(x) &= \sum_{d=0}^{h-1} \sum_{hm+d \leq x} |g[h^2 m + hd + 1] - g[h(hm + d)]| = \\ &= \sum_{d=0}^{h-1} \sum_{hm+d \leq x} \left| \frac{1}{g(Q)} \{g[h^2(Qm + R) + 1] - g[h^2(Qm + R)]\} + \right. \\ &\quad \left. + \frac{g(h)}{g(Q)} \{g[Q(hm + d) + hR - Qd] - g[Q(hm + d)]\} \right|, \end{aligned}$$

and so $T(x) = o(x)$ if $hR - Qd = 0$, because, by using (a), $(h+1, 1) \in \mathbf{J}$ and $h \geq 2$ imply that $(h^2, 1) \in \mathbf{J}$. If $hR - Qd \neq 0$, then we obtain from (16) that

$$0 < hR - Qd = (Q - 1)/h < Q/h,$$

which, by applying (b) with $q = h+1$, implies that $(Q, hR-Qd) \in \mathbf{J}$. This, with $(h^2, 1) \in \mathbf{J}$ shows that $T(x) = o(x)$, i.e. $(h, 1) \in \mathbf{J}$. This completes the proof of (c).

Now we prove Lemma 1.

As we have seen above, $(A, 1) \in \mathbf{J}$. If $A = 1$, then the assertion of Lemma 1 holds. If $A \geq 2$, then by using (c) one can deduce that $(2, 1) \in \mathbf{J}$, and so by applying (b) with $q = 2$, it follows that $(Q, R) \in \mathbf{J}$ for all integers Q and R which satisfy (12). This completes the proof of Lemma 1.

Lemma 2. *Assume that a completely multiplicative function $g(n)$ satisfies the conditions (3) and (4). Then for each positive integer κ , we have*

$$\sum_{n \leq x} |[g(n+1)]^\kappa - [g(n)]^\kappa| = o(x) \quad \text{as } x \rightarrow \infty.$$

PROOF. We first consider the case $\kappa = 1$.

Let $Q \geq 2$ be a fixed positive integer. For each integer $\gamma \geq 0$ let

$$\mathcal{B}_\gamma = \{n \in \mathbf{N} \mid Q^\gamma \parallel (n+1)\}$$

and

$$S_\gamma(x) := x^{-1} \sum_{\substack{n \leq x \\ n \in \mathcal{B}_\gamma}} |g(n+1) - g(n)|.$$

By using the conditions (3) and (4), one can get from Lemma 1 that

$$(17) \quad S_0(x) = x^{-1} \sum_{\substack{n \leq x \\ n \in \mathcal{B}_0}} |g(n+1) - g(n)| = o(1) \quad \text{as } x \rightarrow \infty.$$

Thus, by using (17) and Lemma 1, it follows that

$$\begin{aligned} (18) \quad S_\gamma(x) &:= x^{-1} \sum_{\substack{n \leq x \\ n \in \mathcal{B}_\gamma}} |g(n+1) - g(n)| = \\ &= x^{-1} \sum_{\substack{m+1 \leq (x+1)/Q^\gamma \\ m \in \mathcal{B}_0}} |g(Q^\gamma)g(m+1) - g(Q^\gamma m + Q^\gamma - 1)| = \\ &= x^{-1} \sum_{\substack{m+1 \leq (x+1)/Q^\gamma \\ m \in \mathcal{B}_0}} \left| g(Q^\gamma) [g(m+1) - g(m)] - \right. \\ &\quad \left. - [g(Q^\gamma m + Q^\gamma - 1) - g(Q^\gamma m)] \right| \\ &= o(1). \end{aligned}$$

The relations (17) and (18) together with (3) imply that for each positive integer M , we have

$$\begin{aligned} x^{-1} \sum_{n \leq x} |g(n+1) - g(n)| &\leq S_0(x) + \sum_{1 \leq j \leq M} S_j(x) + x^{-1} \sum_{\substack{n \leq x \\ Q^M | (n+1)}} 2 \leq \\ &\leq o(M+1) + \frac{2}{Q^M}, \end{aligned}$$

and so

$$\limsup_{x \rightarrow \infty} x^{-1} \sum_{n \leq x} |g(n+1) - g(n)| << Q^{-M}.$$

This with $M \rightarrow \infty$ shows that

$$x^{-1} \sum_{n \leq x} |g(n+1) - g(n)| = o(1),$$

which proves Lemma 2 in the case $\kappa = 1$.

Now let $\kappa > 1$ be an integer. By using the relation

$$x^{\kappa+1} - y^{\kappa+1} = x(x^\kappa - y^\kappa) + y^\kappa(x - y),$$

it is easily shown that

$$\mathbf{J}(g) \subseteq \mathbf{J}(g^\kappa) \quad (\kappa = 1, 2, \dots).$$

Thus, Lemma 2 is a consequence of the above relation and the fact $(1, 1) \in \mathbf{J}(g)$. Lemma 2 is proved.

Lemma 3. *Let $f(n)$ be a multiplicative function which satisfies $|f(n)| \leq 1$. Let $1 \leq w_0 \leq x$. Then there is a real number t , $|t| < (\log x)^{1/19}$, so that*

$$\sum_{n \leq x/w} f(n) = w^{-1-it} \sum_{n \leq x} f(n) + O \left[\frac{x}{w} \left(\frac{\log 2w_0}{\log x} \right)^{1/19} \right]$$

uniformly for $1 \leq w \leq w_0$. If f is real-valued, then we may set $t = 0$. The implied constant is absolute.

PROOF. This is Theorem 1 of ELLIOTT [2].

PROOF OF THE THEOREM. Assume that a completely multiplicative function $g(n)$ satisfies the conditions (3), (4) and (5) for some positive integers A , B , k and a non-zero complex number C . Let

$$f(n) := [g(n)]^k \quad (n = 1, 2, \dots).$$

It is obvious that $f(n)$ is a completely multiplicative function, $f(n)$ satisfies (3), furthermore by applying Lemma 2 with $\kappa = k$ it follows that $(1, 1) \in \mathbf{J}(f)$. This shows that $(Q, R) \in \mathbf{J}(f)$ holds for all positive integers Q and R . Let

$$S(x) := \sum_{n \leq x} f(n).$$

Let w_0 be a sufficiently large real number. For each $x > w_0$, by applying Lemma 3, there is a real number $t(x)$ satisfying $|t(x)| \leq (\log x)^{1/19}$ such that for $1 \leq Q \leq w_0$ we have

$$\sum_{m \leq x/Q} f(m) = Q^{-1-it(x)} S(x) + O \left[\frac{x}{Q} \left(\frac{\log 2w_0}{\log x} \right)^{1/19} \right].$$

From this, we have

$$\begin{aligned} \sum_{\substack{n \leq x \\ n \equiv 0 \pmod{Q}}} f(n) &= f(Q) \sum_{m \leq x/Q} f(m) = \\ (19) \quad &= Q^{-1-it(x)} f(Q) S(x) + O \left[\frac{x}{Q} f(Q) \left(\frac{\log 2w_0}{\log x} \right)^{1/19} \right]. \end{aligned}$$

By using (19) and the fact $(Q, R) \in \mathbf{J}(f)$ for all integers Q and R , we deduce that

$$\begin{aligned} \sum_{\substack{n \leq x \\ n \equiv R \pmod{Q}}} f(n) &= \sum_{\substack{n \leq x \\ n \equiv R \pmod{Q}}} [f(n) - f(n-R)] + \sum_{\substack{m \leq x-R \\ m \equiv 0 \pmod{Q}}} f(m) = \\ &= Q^{-1-it(x)} f(Q) S(x) + O \left[\frac{x}{Q} f(Q) \left(\frac{\log 2w_0}{\log x} \right)^{1/19} \right] + o(x) \end{aligned}$$

holds for each $R = 0, \dots, Q-1$. Thus by adding the above relations, we get

$$(20) \quad S(x) = Q^{-it(x)} f(Q) S(x) + O \left[x f(Q) \left(\frac{\log 2w_0}{\log x} \right)^{1/19} \right] + o(Qx).$$

By the condition (5), we can choose $D > 0$ and a sequence $\{x_i\}_{i=1}^{\infty}$, $x_i \rightarrow \infty$ such that

$$\left| \frac{S(x_i)}{x_i} \right| \geq D > 0 \quad \text{as} \quad x_i \rightarrow \infty.$$

Then (20) gives

$$D \left| 1 - \frac{f(Q)}{Q^{it(x_i)}} \right| \leq \left| 1 - \frac{f(Q)}{Q^{it(x_i)}} \right| \cdot \left| \frac{S(x_i)}{x_i} \right| = o(1)$$

and so

$$(21) \quad Q^{it(x_i)} \rightarrow f(Q) \quad \text{as } x_i \rightarrow \infty.$$

Since (21) holds for all integers Q for which $1 \leq Q \leq w_0$, and for each Q we get from (21) that

$$(22) \quad t(x_i) \rightarrow t \quad \text{as } x_i \rightarrow \infty,$$

thus (21) and (22) imply

$$(23) \quad f(Q) = Q^{it}$$

for all $1 \leq Q \leq w_0$. This with $w_0 \rightarrow \infty$ shows that (23) holds for all positive integers Q .

Since

$$f(n) = [g(n)]^k \quad \text{and} \quad f(n) = n^{it} \quad (n = 1, 2, \dots),$$

it follows that for each positive integer n there exists a complex number $G(n)$ such that

$$(24) \quad g(n) = n^{it/k} \cdot G(n).$$

It is obvious that $G(n)$ is a completely multiplicative function and

$$[G(n)]^k = 1 \quad (n = 1, 2, \dots).$$

Let $\tau := t/k$. By (24) we have

$$\begin{aligned} G(An + B) - G(An) &= \frac{g(An + B) - g(An)}{(An)^{i\tau}} - \\ &\quad - G(An + B) \frac{(An + B)^{i\tau} - (An)^{i\tau}}{(An)^{i\tau}} \end{aligned}$$

which with (4) implies that

$$\sum_{n \leq x} |G(An + B) - G(An)| = o(x) \quad \text{as } x \rightarrow \infty.$$

By using Lemma 2 with $g(n)$ replaced by $G(n)$, the last relation implies (8). This completes the proof of our theorem.

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References

- [1] P.D.T.A. ELLIOTT, Probabilistic Number Theory I: Mean-value Theorem, *Grund. der Math. Wiss.* 239, Springer-Verlag, New York, Berlin, 1979.
- [2] P.D.T.A. ELLIOTT, Extrapolating the mean-values of multiplicative functions, *Indagationes Math.* **51** (1989), 409–420.
- [3] A. HILDEBRAND, An Erdős-Wintner theorem for differences of additive functions, *Trans. Amer. Math. Soc.* **310** (1988), 257–276.
- [4] A. HILDEBRAND, Multiplicative functions at consecutive integers I., *Math. Proc. Camb. Phil. Soc.* **100** (1986), 229–236.
- [5] A. HILDEBRAND, Multiplicative functions at consecutive integers II, *Math. Proc. Camb. Phil. Soc.* **103** (1988), 389–398.
- [6] I. KÁTAI, Some problems in number theory, *Studia Sci. Math. Hungar.* **16** (1981), 289–295.
- [7] I. KÁTAI, Multiplicative functions with regularity properties VI, *Acta Math. Acad. Hungar.*, (to appear).
- [8] J.L. MAUCLAIRE AND L. MURATA, On the regularity of arithmetic multiplicative functions I, *Proc. Japan Acad. Ser. A. Math. Sci.* **56** (1980), 438–440.
- [9] B.M. PHONG, A characterization of some arithmetical multiplicative functions, *Acta Math. Acad. Sci. Hungar.*, (to appear).
- [10] B.M. PHONG, On a theorem of Káta-Wirsing, *Acta Sci. Math. Szeged*, (to appear).

BUI MINH PHONG
EÖTVÖS LORÁND UNIVERSITY
COMPUTER CENTER
BOGDÁNFY U. 10/B.
H—1117 BUDAPEST
HUNGARY

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