

Weak topology and Markov–Kakutani theorem on hyperspace

By THAKYIN HU (Taipei) and JUI-CHI HUANG (Taipei)

Abstract. Let K be a weakly compact convex subset of a Banach space X . One version of the Markov–Kakutani Theorem states that if $\mathcal{F} : (K, \tau_w) \rightarrow (K, \tau_w)$ is a commutative family of continuous linear transformations, then $\mathcal{F}$ has a common fixed point in K . Suppose now $CC(X)$ is the collection of all non-empty compact convex subsets of X . We shall define a certain weak topology $\mathcal{J}_w$ on $CC(X)$ and get the above-mentioned version of the Markov–Kakutani Theorem extended to the hyperspace $(CC(X), \mathcal{J}_w)$.

1. Introduction

The classical Markov–Kakutani Theorem states that if K is a compact, convex subset of a topological linear space E , then every commutative family $\mathcal{F}$ of continuous linear transformations of K into K must have a common fixed point in K . Since a linear transformation between Banach spaces is continuous if and only if it is weakly continuous, it follows that if K is a weakly compact convex subset of a Banach space X , then every commutative family $\mathcal{F}$ of weakly continuous linear transformations of K into K must have a common fixed point in K . In this paper, we shall have the above-mentioned version of the Markov–Kakutani theorem extended to the hyperspace $CC(X)$, where $CC(X)$ is the collection of all non-empty compact convex subsets of X .

Mathematics Subject Classification: 47H10, 54H25.

Key words and phrases: fixed point, hyperspace, weak topology.

2. Notations and preliminaries

Let X be a Banach space, X^* its topological dual, and $CC(X)$ the collection of all non-empty compact convex subsets of X . For $A, B \in CC(X)$, define $N(A; \varepsilon) = \{x \in X : \|x - a\| < \varepsilon \text{ for some } a \in A\}$ and $h(A, B) = \inf\{\varepsilon > 0 : A \subset N(B; \varepsilon) \text{ and } B \subset N(A; \varepsilon)\}$ where h is known as the Hausdorff metric induced by the norm on X . Suppose $A, B \in CC(X)$ and α is a scalar, then it is known that both $\lambda A = \{\lambda a : a \in A\}$ and $A + B = \{a + b : a \in A, b \in B\}$ belong to $CC(X)$. Thus $CC(X)$ carries an “affine” structure on it in a natural way. $(CC(X), h)$ is known as the hyperspace over X and it has been investigated by several mathematicians from different view-points ([1], [2], [4]). We now define a subset $\mathcal{K} \subset CC(X)$ to be *convex* if for $A_1, A_2, \dots, A_n \in \mathcal{K}$ and $\alpha_1, \alpha_2, \dots, \alpha_n \in [0, 1]$ with $\sum_{i=1}^n \alpha_i = 1$, we have $\sum_{i=1}^n \alpha_i A_i \in \mathcal{K}$; also a mapping $T : CC(X) \rightarrow CC(X)$ is said to be *affine* if $T(\sum_{i=1}^n \alpha_i A_i) = \sum_{i=1}^n \alpha_i T(A_i)$.

Lemma 1. *Let $A, B, C, D \in CC(X)$. We have*

- (a) $h(\alpha A, \alpha B) = |\alpha| h(A, B)$, where α is a scalar;
- (b) $h(A + C, B + D) \leq h(A, B) + h(C, D)$;
- (c) for each $x^* \in X^*$ and $A_1, A_2, \dots, A_n \in CC(X)$,
 $x^*(\sum_{i=1}^n \alpha_i A_i) = \sum_{i=1}^n \alpha_i x^*(A_i)$,
 where $\alpha_1, \alpha_2, \dots, \alpha_n \in [0, 1]$ with $\sum_{i=1}^n \alpha_i = 1$;
- (d) $A = B$ if and only if $x^*(A) = x^*(B)$ for each $x^* \in X^*$.

The proofs of (a), (b) and (c) follow immediately from the definitions and the proof of (d) is a simple application of the Hahn–Banach theorem and shall be omitted.

Next, we let $\mathbb{Z}$ denote the complex plane, $CC(\mathbb{Z})$ denote the collection of all non-empty compact, convex subsets of $\mathbb{Z}$ and h the natural Hausdorff metric on $CC(\mathbb{Z})$. Note that for any $x^* \in X^*$ and $A \in CC(X)$, it follows from the linearity and continuity of x^* that $x^*(A)$ is a non-empty compact, convex subset of $\mathbb{Z}$, i.e. $x^*(A) \in CC(\mathbb{Z})$. We shall now prove the following

Lemma 2. *Suppose $A, B \in CC(X)$, then $h(x^*(A), x^*(B)) \leq \|x^*\| h(A, B)$ for each $x^* \in X^*$. Thus $x^* : (CC(X), h) \rightarrow (CC(\mathbb{Z}), h)$ is continuous (for simplicity the same h is used to denote different Hausdorff metrics on $CC(X)$ and $CC(\mathbb{Z})$).*

PROOF. Let $r > h(A, B)$. Then $A \subset N(B; r)$ and $B \subset N(A; r)$. Hence for each $a \in A$, there exists $b \in B$ such that $\|a - b\| < r$ and

consequently $\|x^*(a) - x^*(b)\| = \|x^*(a - b)\| \leq \|x^*\| \|a - b\| < \|x^*\| r$, which in turn implies that $x^*(A) \subset N(x^*(B); \|x^*\| r)$. Similarly $x^*(B) \subset N(x^*(A); \|x^*\| r)$. Hence $h(x^*(A), x^*(B)) \leq \|x^*\| r$, which implies that $h(x^*(A), x^*(B)) \leq \|x^*\| h(A, B)$ and the proof is complete.

Recall now that the weak topology τ_w on X is defined to be the weakest topology on X which makes each $x^* : (X, \tau_w) \rightarrow (\mathbb{Z}, | \cdot |)$ continuous. Now that we have, by Lemma 2, that each $x^* : (CC(X), h) \rightarrow (CC(\mathbb{Z}), h)$ is continuous, we may define $\mathcal{J}_w$ to be the weakest topology on the hyperspace $CC(X)$ such that each $x^* : (CC(X), \mathcal{J}_w) \rightarrow (CC(\mathbb{Z}), h)$ is continuous. The notion of weak convergence has been studied by some mathematicians ([3], [5], [6]) and this paper has been inspired by their work; in particular by the paper of F. S. DE BLASI and J. MYJAK [3]. However, our approach is somewhat different from theirs. We shall use the notation $\mathcal{W}(A; x_1^*, \dots, x_n^*; \varepsilon) = \{B \in CC(X) \mid h(x_i^* B, x_i^* A) < \varepsilon \text{ for } i = 1, 2, \dots, n\}$ to denote a $\mathcal{J}_w$ -neighborhood of A in $CC(X)$.

Lemma 3. *Suppose $S, T : (CC(X), \mathcal{J}_w) \rightarrow (CC(X), \mathcal{J}_w)$ are continuous and α is a scalar. Then $S + T : (CC(X), \mathcal{J}_w) \rightarrow (CC(X), \mathcal{J}_w)$ and $\alpha S : (CC(X), \mathcal{J}_w) \rightarrow (CC(X), \mathcal{J}_w)$ are also continuous where $(S + T)(A) = SA + TA$ and $(\alpha S)A = \alpha SA$.*

PROOF. To show that $S + T$ is continuous at A we let $\mathcal{W}(SA + TA; x_1^*, x_2^*, \dots, x_n^*; \varepsilon)$ be a $\mathcal{J}_w$ -neighborhood of $SA + TA$. Since S is continuous at A , for $\mathcal{W}_1(SA; x_1^*, x_2^*, \dots, x_n^*; \frac{\varepsilon}{2})$ which is a $\mathcal{J}_w$ -neighborhood of SA , there exists a $\mathcal{J}_w$ -neighborhood $\mathcal{U}(A)$ such that $B \in \mathcal{U}(A)$ implies $SB \in \mathcal{W}_1$. Similarly, for $\mathcal{W}_2(TA; x_1^*, x_2^*, \dots, x_n^*; \frac{\varepsilon}{2})$, there exists a $\mathcal{J}_w$ -neighborhood $\mathcal{V}(A)$ such that $B \in \mathcal{V}(A)$ implies $TB \in \mathcal{W}_2$. Consequently for $B \in \mathcal{U} \cap \mathcal{V}$ we have $SB \in \mathcal{W}_1$, $TB \in \mathcal{W}_2$ and it follows from Lemma 1 that

$$\begin{aligned} h(x_i^*(SB + TB), x_i^*(SA + TA)) &= h(x_i^*(SB) + x_i^*(TB), \\ &\quad x_i^*(SA) + x_i^*(TA)) \leq h(x_i^*(SB), \\ &\quad x_i^*(SA)) + h(x_i^*(TB), x_i^*(TA)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon; \end{aligned}$$

i.e., $(S + T)(B) \in \mathcal{W}$ proving that $(S + T)$ is continuous at A . It can be proved in a similar fashion, that $\alpha A : (CC(X), \mathcal{J}_w) \rightarrow (CC(X), \mathcal{J}_w)$.

Theorem. *Let X be a Banach space and $\mathcal{K}$ $\mathcal{J}_w$ -compact, convex subset of $(CC(X), \mathcal{J}_w)$. Suppose $\mathcal{F}$ is a commutative family of continuous affine mappings of $(\mathcal{K}, \mathcal{J}_w)$ into itself. Then $\mathcal{F}$ has a common fixed point in $\mathcal{K}$.*

PROOF. For each $T \in \mathcal{F}$ and each integer n , let $T_n = (\frac{1}{n}) \sum_{k=0}^{n-1} T^k$, where $T^0 = I$ is the identity mapping. It follows that $T_n(\mathcal{K}) \subset \mathcal{K}$. It follows from Lemma 3 that T_n is $\mathcal{J}_w$ -continuous and consequently $T_n(\mathcal{K})$ is $\mathcal{J}_w$ -compact. The commutativity of $\mathcal{F}$ implies that $T_n(S_m(\mathcal{K})) = S_m(T_n(\mathcal{K})) \subset T_n(\mathcal{K}) \cap S_m(\mathcal{K})$. Consequently, $\{T_n(\mathcal{K}) : n = 1, 2, \dots, T \in \mathcal{F}\}$ is a family of $\mathcal{J}_w$ -compact subsets of $\mathcal{K}$ with finite intersection property and hence has non-empty intersection, i.e., $\bigcap T_n(\mathcal{K}) \neq \emptyset$ where the intersection is taken over $n = 1, 2, \dots$, and $T \in \mathcal{F}$. Let $A_0 \in \bigcap T_n(\mathcal{K})$. We claim that $TA_0 = A_0$ for all $T \in \mathcal{F}$. Assume the contrary, then there exists $T \in \mathcal{F}$ with $TA_0 \neq A_0$ which implies that $h(x^*(TA_0), x^*(A_0)) > 0$ for some $x^* \in X^*$ by Lemma 1. For each n , $A_0 \in T_n(\mathcal{K})$ implies the existence of some $B_n \in \mathcal{K}$ with $A_0 = \frac{1}{n} \sum_{k=0}^{n-1} T^k(B_n)$. T is affine implies that $TA_0 = (\frac{1}{n}) \sum_{k=1}^n T^k(B_n)$. Since $x^* : (CC(X), \mathcal{J}_w) \rightarrow (CC(\mathbb{Z}), h)$ is continuous and $\mathcal{K}$ is $\mathcal{J}_w$ -compact, it follows that $x^*(\mathcal{K})$ is a compact subset of the metric space $(CC(\mathbb{Z}), h)$ and hence totally bounded which in turn implies that $\text{diam}(\mathcal{K}) = \sup\{h(x^*(A), x^*(B)) : A, B \in \mathcal{K}\} < \infty$. It follows now from the lemmas that

$$\begin{aligned} h(x^*(TA_0), x^*(A_0)) &= h\left(x^*\left(\frac{1}{n} \sum_{k=1}^n T^k B_n\right), x^*\left(\frac{1}{n} \sum_{k=0}^{n-1} T^k B_n\right)\right) = \\ &\quad \frac{1}{n} h(x^*(TB_n) + x^*(T^2 B_n) + \dots + x^*(T^n B_n), \\ &\quad x^*(B_n) + x^*(TB_n) + \dots + x^*(T^{n-1} B_n)) \leq \frac{1}{n} h(x^*(B_n), x^*(T^n B_n)) \\ &\leq \frac{1}{n} \text{diam}(\mathcal{K}). \end{aligned}$$

$\text{Diam}(\mathcal{K}) < \infty$ and that n is arbitrary implies that $h(x^*(TA_0), x^*(A_0)) = 0$. This is a contradiction. Thus $TA_0 = A_0$ for all $T \in \mathcal{F}$ and the proof is complete.

Remark. Suppose $\mathcal{K}$ consists of singletons. Then we obtain the version of the Markov–Kakutani Theorem mentioned in the introduction of this paper.

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THAKYIN HU
DEPARTMENT OF MATHEMATICS
TAMKANG UNIVERSITY, TAMSUI
TAIPEI, TAIWAN 25137
REPUBLIC OF CHINA

JUI-CHI HUANG DEPARTMENT OF MATHEMATICS
TAMKANG UNIVERSITY, TAMSUI
TAIPEI, TAIWAN 25137
REPUBLIC OF CHINA

(Received March 19, 1997)