

On the Cartan connection for a class of generalized Lagrange spaces

By D. HRIMIUC (Edmonton)

Abstract. In this paper we determine the Cartan connection of the generalized Lagrange space (M, a_{ij}) with metric tensor $a_{ij} = \theta(x, y) \overset{\circ}{h}_{ij} + \psi(F^2(x, y)) e^{2\alpha(x)} \overset{\circ}{l}_i \overset{\circ}{l}_j$, which includes particular metrics frequently encountered in applications in Physical and Biological problems. The results obtained are employed on the problem of equivalence of geodesics for certain metrics.

Introduction

The geometrical theory of spaces endowed with metrics having the forms

$$(i) \quad a_{ij} = e^{2\alpha} g_{ij} \quad \text{or} \quad (ii) \quad \bar{g}_{ij} = g_{ij} + \beta \overset{\circ}{y}_i \overset{\circ}{y}_j$$

where g_{ij} is the metric tensor of a Finsler manifold and α, β are functions of position and direction has been studied in many papers [15], [19], [2], [4], [5], [12], [13], [14], [20], [21], [22]. Of course, these spaces used as geometrical models for gravitation and electromagnetism generally are not Lagrange manifolds and new geometrical ideas, initiated in [18] have been used. It is clear that such metrics are useful for a constructive axiomatic theory of general relativity based on conditions formulated by EHLERS, PIRANI and SCHILD ([10], [19]), as well as for a geometrical model construction of relativistic optic [7], [8], [9], [15], [18], [19], [20], [21]. It seems that these metrics can be successfully used for biological models as well,

Mathematics Subject Classification: 53B40, 53C60, 53C05.

Key words and phrases: Finsler spaces, generalized Lagrange spaces, Cartan connection. Partially supported by NSERC Grant A-7667 and by a grant from Central Research Fund at the University of Alberta.

as already given in the particular case of Finsler manifolds [3], or more general for some special Lagrangians [2], [4], [5].

The main result is given in Section 3 where the Cartan connection for a special GL-space is obtained. This is a substantial extension of similar results from Riemannian geometry (Levi-Civita connection) and Finsler geometry (Cartan connection).

The Cartan connection was obtained in [2] for homogeneous case and more general for φ -Lagrangians (different proof) [4], [5]. However, for these GL-spaces the obvious modifications of standard techniques (e.g. Finsler, m -homogeneous, φ -Lagrange) do not work.

The canonical d -connection used for the geometrical study of generalized Lagrange manifolds with metrics of type (i) or (ii) have in general nonvanishing deflection tensor. This is an unpleasant situation from at least two points of view: the energy functional is nonconservative along the paths of nonlinear connection, and the horizontal lift of these paths are not h -paths. In the present work, making a conformal change of metric (ii), we will find other metrics without the inconveniences alluded to above.

1. Preliminaries

Let M be a C^∞ -real n -dimensional differentiable manifold and $\pi : TM \rightarrow M$ its tangent bundle.

If (x^i) , $i = 1, \dots, n$ is a local system of coordinates on a domain U of a chart on M then (x^i, y^i) is the induced local system of coordinates on $\pi^{-1}(U)$ in TM .

We put $\partial_i := \partial/\partial x^i$ and $\dot{\partial}_i := \partial/\partial y^i$.

A change of coordinates on TM is given by

$$(1.1) \quad x^{i'} = x^{i'}(x^i), \quad y^{i'} = (\partial_i x^{i'})y^i, \quad \text{rank}(\partial_i x^{i'}) = n.$$

A *nonlinear connection* on M is defined by a distribution HTM over TM supplementary to the vertical distribution i.e. the kernel of the differential of π .

In HTM there exists a local frame defined by the local vector fields

$$(1.2) \quad \delta_i := \partial_i - N_i^k \dot{\partial}_k$$

(N_i^k) are called *the coefficients of the nonlinear connection*. We can introduce some special tensor fields, called *d-tensor field* as an object of algebra spanned by $\{1, \delta_i, \dot{\partial}_i\}$ over the ring $\mathcal{F}(TM)$ of smooth real valued functions on TM .

For a change of coordinates given by (1.1) the components of a *d*-tensor are transformed in exactly the same way as a tensor on M , in spite of y^i dependence.

Definition 1.1 ([18]). A *generalized Lagrange space* (briefly *GL-space*) is a pair $(M, a_{ij}(x, y))$ where a_{ij} is a symmetric and nondegenerate *d*-tensor field on $\overset{\circ}{TM} = TM \setminus \{0_x, x \in M\}$.

If there exists a smooth function $L : TM \rightarrow \mathbb{R}$ such that $a_{ij} = 1/2 \dot{\partial}_i \dot{\partial}_j L$, then the pair (M, a_{ij}) or (M, L) is called a Lagrange space.

A Finsler space is a particular Lagrange space and of course, a particular *GL-space*. In this case L is smooth on $\overset{\circ}{TM}$, positively homogeneous and the quadratic form $a_{ij}(x, y)$ is positive definite.

The geometry of *GL-space* can be developed by the same methods as those employed in Lagrange or Finsler space [18], [6], [16].

In a *GL-space* endowed with the nonlinear connection (N_j^i) we can introduce so called *d-connection*.

$CT(N_j^i) = (L_{jk}^i, C_{jk}^i)$, which allows us to define the *h*- and *v*-covariant derivative like for Finsler spaces. For example, the *h*- and *v*-covariant derivative of the metric tensor a_{ij} , denoted by “ $|$ ” and respectively “ $|$ ” are given by

$$a_{ij|k} = \delta_k a_{ij} - L_{ki}^s a_{sj} - L_{kj}^s a_{is}; \quad a_{ij} | _k = \dot{\partial}_k a_{ij} - C_{ki}^s a_{sj} - C_{kj}^s a_{is}.$$

The following result (see [18], page 186) will be used:

Theorem 1.2. Let (M, a_{ij}) be a *GL-Lagrange space* and (N_j^i) a fixed nonlinear connection. Then there exist only one *d-connection* $CT(N_j^i) = (L_{jk}^i, C_{jk}^i)$ with the following properties

$$\begin{aligned} 1^\circ \quad a_{ij|k} &= 0 \quad (h\text{-metrical}), & 2^\circ \quad a_{ij} | _k &= 0 \quad (v\text{-metrical}), \\ 3^\circ \quad T_{jk}^i &= L_{jk}^i - L_{kj}^i = 0, & 4^\circ \quad S_{jk}^i &= C_{jk}^i - C_{kj}^i = 0. \end{aligned}$$

The coefficients of this d -connection are given by

$$\begin{aligned} L_{jk}^i &= \frac{1}{2} a^{ip} (\delta_j a_{pk} + \delta_k a_{jp} - \delta_p a_{jk}) \\ C_{jk}^i &= \frac{1}{2} a^{ip} (\dot{\partial}_j a_{pk} + \dot{\partial}_k a_{jp} - \dot{\partial}_p a_{jk}) \end{aligned}$$

This connection will be called *the canonical d -connection* of the GL -space (M, a_{ij}) .

A systematic presentation of the geometry of a GL -space is given in [18]. Also, throughout this paper we shall use the usual set up for general theory of Finsler manifolds. We follow closely the exposition of basic Finsler geometry in [6], [16].

2. Deflection free connection on GL -spaces

Let $C\overset{\circ}{\Gamma} = (\overset{\circ}{N}_j^i, \overset{\circ}{F}_{jk}^i, \overset{\circ}{C}_{jk}^i)$ be the Cartan connection of the Finsler space $F^n = (M, F(x, y))$ and $g_{ij} := 1/2 \dot{\partial}_i \dot{\partial}_j F^2$ the metric tensor. In general, the geometrical objects related to the Finsler space will be indicated by “0” on the top.

For example $\overset{\circ}{y}_i = g_{ij} y^j$, $\overset{\circ}{\ell}_i = \overset{\circ}{y}_i / F$, $\overset{\circ}{\ell}^i = y^i / F$, $\overset{\circ}{h}_{ij} = g_{ij} - \overset{\circ}{\ell}_i \overset{\circ}{\ell}_j$. We consider a GL -space whose metric tensor is of the form

$$(2.1) \quad a_{ij} = e^{2\alpha} \bar{g}_{ij}, \quad \bar{g}_{ij} = g_{ij} + \beta \overset{\circ}{y}_i \overset{\circ}{y}_j,$$

where $\alpha = \alpha(x, y)$, $\beta = \beta(x, y)$ are smooth scalar functions such that $1 + \beta F^2 > 0$. It is easy to see that the reciprocal components of a_{ij} are given by

$$(2.2) \quad a^{ij} = e^{-2\alpha} \bar{g}^{ij}, \quad \bar{g}^{ij} = g^{ij} - \frac{\beta}{1 + F^2 \beta} y^i y^j.$$

We put

$$(2.3) \quad U_{jk}^i := \delta_k^i \alpha_j + \delta_j^i \alpha_k - \bar{g}_{jk} \bar{g}^{ih} \alpha_h + \frac{1}{2} \bar{g}^{ih} (\beta_j \overset{\circ}{y}_h \overset{\circ}{y}_k + \beta_k \overset{\circ}{y}_j \overset{\circ}{y}_h - \beta_h \overset{\circ}{y}_j \overset{\circ}{y}_k)$$

$$(2.4) \quad \dot{U}_{jk}^i := \delta_k^i \dot{\alpha}_j + \delta_j^i \dot{\alpha}_k - \bar{g}_{jk} \bar{g}^{ih} \dot{\alpha}_h + \frac{1}{2} \bar{g}^{ih} (\dot{\beta}_j \overset{\circ}{y}_h \overset{\circ}{y}_k + \dot{\beta}_k \overset{\circ}{y}_j \overset{\circ}{y}_h - \dot{\beta}_h \overset{\circ}{y}_j \overset{\circ}{y}_k)$$

where $\alpha_i := \delta_i \alpha$, $\dot{\alpha}_i := \dot{\partial}_i \alpha$. We have

Proposition 2.1. Let $\overset{\circ}{CT} = (\overset{\circ}{N}_j^i, F_{jk}^i, \overset{\circ}{C}_{jk}^i)$ be the Cartan connection of the Finsler space $F^n = (M, F(x, y))$. The coefficients of the canonical d -connection $\overset{\circ}{CT}(\overset{\circ}{N}_j^i) = (L_{jk}^i, C_{jk}^i)$ of the GL -space (M, a_{ij}) are given by

$$(2.5) \quad L_{jk}^i = F_{jk}^i + U_{jk}^i$$

$$(2.6) \quad C_{jk}^i = \overset{\circ}{C}_{jk}^i + \dot{U}_{jk}^i + \frac{\beta}{1 + \beta F^2} g_{jk} y^i.$$

PROOF. We can apply Theorem 1.2 or we look for L_{jk}^i in the following form:

$$L_{jk}^i = F_{jk}^i + A_{jk}^i$$

where A_{jk}^i is a symmetric d -tensor field.

From $a_{ij|k} = 0 \iff a_{ij} \overset{\circ}{|}_k - a_{hj} A_{ik}^h - a_{ih} A_{jk}^h = 0$ using the Christoffel process, after a tedious calculation we obtain (2.5). Similarly, we get (2.6).

For the d -connection $\overset{\circ}{CT}(\overset{\circ}{N}_j^i) = (L_{jk}^i, C_{jk}^i)$ we put the following questions:

- a) When the horizontal lift of the paths of the nonlinear connection $(\overset{\circ}{N}_j^i)$, (in fact the geodesics of the Finsler spaces, parametrized by arc length) are h -paths? (This is equivalent with $D_0^i = L_{00}^i$.)
- b) For which α and β the deflection tensor, D_j^i , vanishes?

Theorem 2.2. Let $\overset{\circ}{CT}(\overset{\circ}{N}_j^i) = (L_{jk}^i, C_{jk}^i)$ the canonical d -connection given by (2.5), (2.6) and $\mathcal{E} = a_{ij} y^i y^j$ the energy function. Then

- (i) $\mathcal{E}_{|k} = 0 \iff 2\alpha = \sigma - \log(1 + F^2\beta)$ with $\sigma_k = 0 \iff D_0^i = 0$
- (ii) $D_k^i = 0 \iff 2\alpha = \sigma - \log(1 + F^2\beta)$ with $\sigma_k = 0$ and $\beta_0 := y^i \beta_i = 0$.

PROOF. (i) The h -covariant derivatie of $\mathcal{E} = e^{2\alpha}(1 + \beta F^2)F^2$ is

$$(2.7) \quad \mathcal{E}_{|k} = e^{2\alpha} F^2 (2(1 + \beta F^2)\alpha_k + F^2 \beta_k).$$

Thus

$$\mathcal{E}_{|k} = 0 \iff 2\alpha_k = -\frac{\beta_k F^2}{1 + \beta F^2} \iff 2\alpha = \sigma - \log(1 + \beta F^2)$$

where $\sigma = \sigma(x, y)$ and $\sigma_k = 0$.

On the other hand

$$D_j^i = L_{j0}^i - \overset{\circ}{N}_j^i = U_{j0}^i.$$

That is

$$\bar{g}_{ih} D_j^h = (\bar{g}_{ij} \alpha_k + \bar{g}_{ik} \alpha_j - \bar{g}_{jk} \alpha_i) y^k + \frac{1}{2} (\beta_k \overset{\circ}{y}_i \overset{\circ}{y}_j + \beta_j \overset{\circ}{y}_i \overset{\circ}{y}_k - \beta_i \overset{\circ}{y}_j \overset{\circ}{y}_k) y^k.$$

Taking (2.1) into account we get

$$(2.8) \quad \begin{aligned} \bar{g}_{ih} D_j^h &= ((\beta \alpha_k + \frac{1}{2} \beta_k) \overset{\circ}{y}_i \overset{\circ}{y}_j + (\beta \alpha_j + \frac{1}{2} \beta_j) \overset{\circ}{y}_i \overset{\circ}{y}_k \\ &\quad - (\beta \alpha_i + \frac{1}{2} \beta_i) \overset{\circ}{y}_j \overset{\circ}{y}_k) y^k \\ &\quad + (g_{ij} \alpha_k + g_{ik} \alpha_j - g_{jk} \alpha_i) y^k. \end{aligned}$$

Using $\alpha_k = -\frac{F^2 \beta_k}{2(1+\beta F^2)}$ we obtain after same computations

$$(2.9) \quad \bar{g}_{ih} D_j^h = \frac{\beta_0}{2(1+\beta F^2)} (\overset{\circ}{y}_i \overset{\circ}{y}_j - F^2 g_{ij}).$$

Transvecting by y^j in (2.8) we get $D_0^i = 0$. Now, suppose that $D_0^i = 0$. From (2.8) we have

$$(2.10) \quad 2\alpha_0 \overset{\circ}{y}_i - \alpha_i F^2 + (2\beta \alpha_0 + \beta_0) F^2 \overset{\circ}{y}_i - (\beta \alpha_i + \frac{1}{2} \beta_i) F^4 = 0$$

Transvecting by y^i we get

$$2\alpha_0 = -\frac{\beta_0 F^2}{1+\beta F^2}.$$

Using again (2.10) we obtain

$$(2.11) \quad 2\alpha_k = -\frac{\beta_k F^2}{1+\beta F^2}$$

and (i) follows.

(ii) If $D_k^i = 0$ then $D_0^i = 0$ and using (i) we get $2\alpha = \sigma - \log(1 + F^2 \beta)$ with $\sigma_k = 0$. On the other hand making use of (2.9) we obtain $\beta_0 = 0$. The converse implication follows from (2.8) and (2.9).

Corollary 2.3. *In the GL -space with metric tensor (2.1) horizontal lift of the paths of the nonlinear connection $\overset{\circ}{N}_j^i$ (the same with the geodesics of $F^n = (M, F(x, y))$ parametrized by arc length) are h -paths of $CT(\overset{\circ}{N}_j^i) = (L_{jk}^i, C_{jk}^i)$ iff $2\alpha_k = \frac{\beta_k F^2}{1+\beta F^2}$.*

From the Theorem 2.2 is clear that starting with the Cartan connection $\overset{\circ}{CT}$ of the Finsler space $F^n = (M, F(x, y))$ the deflection tensor of the canonical d -connection of GL -space (M, a_{ij}) vanishes iff

$$(2.12) \quad a_{ij} = e^{\sigma - \log(1+F^2)\beta} (g_{ij} + \beta \overset{\circ}{y}_i \overset{\circ}{y}_j)$$

where $\sigma = \sigma(x, y)$ with $\sigma_k = 0$. We can write (2.12) after changing notation to the following form

$$(2.13) \quad a_{ij} = e^\sigma (\theta g_{ij} + (1 - \theta) \overset{\circ}{\ell}_i \overset{\circ}{\ell}_j)$$

where $\theta = \theta(x, y)$, $\theta \neq 0$ and $\sigma_k = 0$.

These results have suggested we give up use at the start of the Cartan nonlinear connection $\overset{\circ}{N}_j^i$ and look for the Cartan connection of more general metrics than (2.13).

Remark. For the GL -manifold $(M, \bar{g}_{ij})$ the deflection tensor of $CT(\overset{\circ}{N}_j^i) = (L_{jk}^i, C_{jk}^i)$ is null iff $\beta_k = 0$. Then a conformal change of this metric leads to a new GL -space for which the horizontal lift of the geodesics of its associated Finsler space (parametrized by arc length) are h -paths, in less restrictive conditions.

3. The Cartan connection for some special GL -space

We consider the GL -space (M, a_{ij}) with metric tensor given by

$$(3.1) \quad a_{ij} = \theta g_{ij} + (\psi(F^2)e^{2\alpha} - \theta) \overset{\circ}{\ell}_i \overset{\circ}{\ell}_j$$

where

$$\theta = \theta(x, y), \quad \alpha = \alpha(x) \quad \text{are smooth scalar functions,} \quad \theta \neq 0,$$

and $\psi : (0, \infty) \rightarrow (0, \infty)$ is a smooth function such that $\psi(t) + t\psi'(t) \neq 0$ $\forall t \in (0, \infty)$.

It is easy to check that the reciprocal components of a_{ij} are given by

$$(3.2) \quad a^{ij} = \frac{1}{\theta} g^{ij} + \left(\frac{1}{\psi} e^{-2\alpha} - \frac{1}{\theta} \right) \overset{\circ}{\ell}^i \overset{\circ}{\ell}^j.$$

Using the angular metric tensor $\overset{\circ}{h}_{ij} = g_{ij} - \overset{\circ}{\ell}_i \overset{\circ}{\ell}_j$ we can rewrite (3.1) and (3.2) in the following simple forms:

$$(3.1)' \quad a_{ij} = \theta \overset{\circ}{h}_{ij} + \psi e^{2\alpha} \overset{\circ}{\ell}_i \overset{\circ}{\ell}_j$$

$$(3.2)' \quad a^{ij} = \frac{1}{\theta} \overset{\circ}{h}^{ij} + \frac{1}{\psi} e^{-2\alpha} \overset{\circ}{\ell}^i \overset{\circ}{\ell}^j$$

where $\overset{\circ}{h}^{ij} = g^{ip} g^{jq} \overset{\circ}{h}_{pq} = g^{ij} - \overset{\circ}{\ell}^i \overset{\circ}{\ell}^j$.

The metric tensor a_{ij} is quite general and many other particular metrics used in different papers can be obtained from it.

Examples. 1. For $\psi = 1$, $\alpha = 0$, $\theta = c$, $g_{ij}(x, y) = \gamma_{ij}(x)$ (Riemannian metric)

$$(3.3) \quad a_{ij} = c\gamma_{ij} + (1 - c) \overset{\circ}{\ell}_i \overset{\circ}{\ell}_j$$

which has been used in [7], [8], [21] in problems of post-Newtonian estimation, and also in [22].

2. For $\psi(t) = \frac{1}{c^2} t + 1$ ($c \in \mathbb{R}^*$), $\alpha = 0$, $\theta = 1$ the metric tensor is given by

$$(3.4) \quad a_{ij} = g_{ij} + \frac{1}{c^2} \overset{\circ}{y}_i \overset{\circ}{y}_j$$

which have been used in [15], [9], [21], [18] for applications in relativistic geometric optics.

3. For $\alpha = 0$, $\theta(x, y) = \varphi'(F^2(x, y))$, $\psi = 2\varphi''(t)t + \varphi'(t)$ we obtain

$$(3.5) \quad a_{ij} = \varphi' g_{ij} + 2\varphi'' \overset{\circ}{y}_i \overset{\circ}{y}_j$$

which is nothing but the metric tensor of a φ -Lagrange [3], [4] which have been used for applications in Biology and Physics.

In particular for $\varphi(t) = t^{m/2}$ we get

$$(3.6) \quad a_{ij} = \frac{m}{2} F^{m-2} (g_{ij} + (m-2) \overset{\circ}{\ell}_i \overset{\circ}{\ell}_j)$$

which is the metric tensor for m -homogeneous Lagrange space [2], [12], [13].

4. If $\theta = 1$, $\psi = 1$, $2\alpha(x) = \ln(1 - \bar{\alpha}(x))$ thus

$$(3.7) \quad a_{ij} = g_{ij} - \bar{\alpha}(x) \overset{\circ}{\ell}_i \overset{\circ}{\ell}_j$$

and we get the GL -space studied in [14].

5. For $\theta = \bar{\theta}(x, y) e^{2\alpha(x)}$, $\bar{\theta} \neq 0$

$$(3.8) \quad a_{ij} = e^{2\alpha(x)} [\bar{\theta} g_{ij} + (\psi - \bar{\theta}) \overset{\circ}{\ell}_i \overset{\circ}{\ell}_j]$$

which is a conformal change of metric $\bar{g}_{ij} = \bar{\theta} g_{ij} + (\psi - \bar{\theta}) \overset{\circ}{\ell}_i \overset{\circ}{\ell}_j$ more general as in [11].

The central problem of this paper is to determine the Cartan connection of the GL -space (M, a_{ij}) where a_{ij} is given by (3.1) or (3.1)'.

Following [6], [16] the Cartan connection is a triple $CT = (N_j^i, L_{jk}^i, C_{jk}^i)$ which must verify the following axioms (Matsumoto's axioms).

$$\begin{aligned} 1^\circ \quad a_{ij}|_k &= 0 & 2^\circ \quad a_{ij}|_k &= 0 \\ 3^\circ \quad D_j^i &= L_{j0}^i - N_j^i = 0 & 4^\circ \quad T_{jk}^i &= L_{jk}^i - L_{kj}^i = 0 \\ 5^\circ \quad S_{jk}^i &= C_{jk}^i - C_{kj}^i = 0. \end{aligned}$$

We shall use the following notations:

$$(3.9) \quad y_i := a_{ij} y^j, \quad \|y\|^2 := y_i y^i = a_{ij} y^i y^j = \mathcal{E}$$

$$(3.10) \quad \ell_i := y_i / \|y\|,$$

$$\ell^i := y^i / \|y\| \quad (\text{normalized supporting element})$$

$$(3.11) \quad h_{ij} := a_{ij} - \ell_i \ell_j \quad (\text{angular metric tensor}).$$

The following properties are immediate

$$(3.12) \quad y_i = \varphi e^{2\alpha} \overset{\circ}{y}_i; \quad \|y\|^2 = \psi F^2 e^{2\alpha}$$

$$(3.13) \quad \ell_i = a_{ij} \ell^j; \quad \ell_i \ell_j = \psi e^{2\alpha} \overset{\circ}{\ell}_i \overset{\circ}{\ell}_j$$

$$(3.14) \quad h_{ij} = \theta \overset{\circ}{h}_{ij}; \quad h_j^i := a^{ik} h_{kj} = \overset{\circ}{h}_j^i.$$

Theorem 3.1. *In the GL-space (M, a_{ij}) with a_{ij} given by (3.1) there exists a unique d-connection $CT = (N_j^i, L_{jk}^i, C_{jk}^i)$ which satisfies the Matsumoto's axioms.*

PROOF. We look for CT which verifies $1^\circ - 5^\circ$. From 1° and 3° we get $\|y\|_k^2 = 0$ and making use of (3.12) we obtain

$$(3.15) \quad F_{|k}^2 = \frac{-2\alpha_k \psi F^2}{\psi + \psi' F^2},$$

where $\alpha_k := \partial_k \alpha$.

From (3.1) using (3.13) we get the following expression for the metric tensor of F^n

$$(3.16) \quad g_{ij} = \frac{1}{\theta} a_{ij} + \left(\frac{e^{-2\alpha}}{\psi} - \frac{1}{\theta} \right) \ell_i \ell_j.$$

We have, making use of 1° and 3°

$$\begin{aligned} g_{ij|k} &= -\frac{\theta_k}{\theta} a_{ij} + \frac{\theta_k}{\theta^2} \ell_i \ell_j - \frac{2e^{-2\alpha}}{\psi} \alpha_k \ell_i \ell_j - \frac{\psi' e^{-2\alpha}}{\psi^2} F_{|k}^2 \ell_i \ell_j \\ &= -\frac{\theta_k}{\theta^2} h_{ij} - \frac{2e^{-2\alpha}}{\psi} \alpha_k \ell_i \ell_j + \frac{\psi' e^{-2\alpha}}{\psi^2} \frac{2\psi F^2}{\psi + \psi' F^2} \alpha_k \ell_i \ell_j. \end{aligned}$$

From (3.13), (3.14) and the equality above it follows

$$g_{ij|k} = -\frac{\theta_k}{\theta} \overset{\circ}{h}_{ij} - 2\alpha_k \overset{\circ}{\ell}_i \overset{\circ}{\ell}_j + \frac{2F^2 \psi'}{\psi + F^2 \psi'} \alpha_k \overset{\circ}{\ell}_i \overset{\circ}{\ell}_j.$$

Therefore

$$(3.17) \quad L_{ik}^p g_{pj} + L_{jk}^p g_{ip} = \delta_k g_{ij} + \frac{1}{\theta} \theta_k \overset{\circ}{h}_{ij} + \frac{2}{1+A} \alpha_k \overset{\circ}{\ell}_i \overset{\circ}{\ell}_j$$

where we have put

$$(3.18) \quad A := \frac{\psi'(F^2)}{\psi(F^2)} F^2.$$

Using the Christoffel process and 4° we get

$$(3.19) \quad \begin{aligned} L_{jk}^i &= \frac{1}{2} g^{ip} (\delta_j g_{pk} + \delta_k g_{jp} - \delta_p g_{jk}) \\ &+ \frac{1}{2\theta} g^{ip} (\theta_j \overset{\circ}{h}_{pk} + \theta_k \overset{\circ}{h}_{jp} - \theta_p \overset{\circ}{h}_{jk}) \\ &+ \frac{1}{1+A} (\alpha_j \overset{\circ}{\ell}_k \overset{\circ}{\ell}^i + \alpha_k \overset{\circ}{\ell}_j \overset{\circ}{\ell}^i - \overset{\circ}{\ell}_j \overset{\circ}{\ell}_k \alpha^i). \end{aligned}$$

Introducing the generalized Christoffel symbols

$$\gamma_{jk}^i = \frac{1}{2} g^{ip} (\partial_j g_{pk} + \partial_k g_{jp} - \partial_p g_{jk})$$

(3.19) can be rewritten in the following form

$$(3.20) \quad \begin{aligned} L_{jk}^i &= \gamma_{jk}^i - N_j^r \overset{\circ}{C}_{rk}^i - N_k^r \overset{\circ}{C}_{rj}^i + g^{ip} N_p^r \overset{\circ}{C}_{rjk} \\ &+ \frac{1}{2\theta} (\theta_j \overset{\circ}{h}_k^i + \theta_k \overset{\circ}{h}_j^i - \theta^i \overset{\circ}{h}_{jk}) \\ &+ \frac{1}{1+A} (\alpha_j \overset{\circ}{\ell}_k \overset{\circ}{\ell}^i + \alpha_k \overset{\circ}{\ell}_j \overset{\circ}{\ell}^i - \overset{\circ}{\ell}_j \overset{\circ}{\ell}_k \alpha^i) \end{aligned}$$

with $\theta^i := g^{ip} \theta_p$.

Transvecting by y^k and then by y^j and using again 3° we get

$$(3.21) \quad \begin{aligned} N_j^i &= \gamma_{j0}^i - N_0^r \overset{\circ}{C}_{rj}^i + \frac{1}{2\theta} \theta_0 \overset{\circ}{h}_j^i \\ &+ \frac{1}{1+A} (F \alpha_j \overset{\circ}{\ell}^i + \alpha_0 \overset{\circ}{\ell}_j \overset{\circ}{\ell}^i - F \overset{\circ}{\ell}_j \alpha^i) \end{aligned}$$

$$(3.22) \quad N_0^i = \gamma_{00}^i + \frac{1}{1+A} (2F \alpha_0 \overset{\circ}{\ell}^i - F^2 \alpha^i)$$

where $\alpha^i := g^{ij} \alpha_j$ and

$$(3.23) \quad \theta_0 = y^k (\partial_k \theta - N_k^i \dot{\theta}_i) = \partial_0 \theta - \gamma_{00}^i \dot{\theta}_i - \frac{1}{1+A} (2\alpha_0 \dot{\theta}_0 - F^2 \alpha^i \dot{\theta}_i).$$

Therefore, making use of (3.22) a direct calculation gives

$$\begin{aligned} N_j^i &= \gamma_{j0}^i - \overset{\circ}{C}_{jk}^i \gamma_{00}^k + \frac{F^2}{1+A} \overset{\circ}{C}_{jk}^i \alpha^k \\ &+ \frac{1}{1+A} \left(\alpha_j y^i + \frac{1}{F^2} \alpha_0 \overset{\circ}{y}_j y^i - \overset{\circ}{y}_j \alpha^i \right) + \frac{\theta_0}{2\theta} \overset{\circ}{h}_j^i. \end{aligned}$$

That is

(3.24)

$$\begin{aligned} N_j^i &= \overset{\circ}{N}_j^i + \frac{1}{1+A} (\delta_j^i \alpha_0 + \alpha_j y^i - \overset{\circ}{y}_j \alpha^i + F^2 \overset{\circ}{C}_{jk}^i \alpha^k) \\ &+ \left(\frac{\theta_0}{2\theta} - \frac{\alpha_0}{1+A} \right) \overset{\circ}{h}_j^i \end{aligned}$$

where

$$\overset{\circ}{N}_j^i = \gamma_{j0}^i - \overset{\circ}{C}_{jk}^i \gamma_{00}^k$$

is the Cartan nonlinear connection of the Finsler space $F^n = (M, F(x, y))$.

L_{jk}^i given by (3.19) with (N_j^i) from (3.24) verifies $1^\circ, 3^\circ, 4^\circ$.

Now, from 2° and 5° we get by standard calculation

$$C_{jk}^i = \frac{1}{2} a^{ik} (\dot{\partial}_j a_{pk} + \dot{\partial}_k a_{jp} - \dot{\partial}_p a_{jk}).$$

Remark. 1. $B_j^i := \delta_j^i \alpha_0 + \alpha_j y^i - \overset{\circ}{y}_j \alpha^i + F^2 \overset{\circ}{C}_{jk}^i \alpha^k$ from (3.24) is just the tensor which perturbs the Cartan nonlinear connection of a Finsler space after a conformal change of metric tensor (see [10]).

In fact, this theorem gives us the change of the Cartan connection $C\overset{\circ}{\Gamma} = (\overset{\circ}{N}_j^i, L_{jk}^i, \overset{\circ}{C}_{jk}^i)$ after a distorted conformal change of the metric tensor g_{ij}

$$g_{ij} \rightarrow \theta g_{ij} + (\psi - \theta) \overset{\circ}{\ell}_i \overset{\circ}{\ell}_j \rightarrow e^{\alpha(x)} (\theta g_{ij} + (\psi - \theta) \overset{\circ}{\ell}_i \overset{\circ}{\ell}_j).$$

2. The d -connection $C\Gamma = (N_j^i, L_{jk}^i, C_{jk}^i)$ has the coefficients given as follows

$$(3.25) \quad L_{jk}^i = \frac{1}{2} a^{ip} (\delta_j a_{pk} + \delta_k a_{jp} - \delta_p a_{jk})$$

$$(3.25)' \quad C_{jk}^i = \frac{1}{2} a^{ip} (\dot{\partial}_j a_{pk} + \dot{\partial}_k a_{jp} - \partial_p a_{jk})$$

with N_j^i from (3.24).

We can also express these coefficients in terms of the Cartan connection of the Finsler space $(M, F(x, y))$.

3. The following formulae hold:

- i) $g_{ij|k} = -\frac{\theta_k}{\theta} \overset{\circ}{h}_{ij} - \frac{2}{1+A} \alpha_k \overset{\circ}{\ell}_i \overset{\circ}{\ell}_j$
- ii) $\overset{\circ}{\ell}_{i|k} = -\frac{1}{1+A} \alpha_k \overset{\circ}{\ell}_i$, $\overset{\circ}{\ell}_{|k}^i = \frac{1}{1+A} \alpha_k \overset{\circ}{\ell}^i$
- iii) $\overset{\circ}{h}_{ij|k} = -\frac{\theta_k}{\theta} \overset{\circ}{h}_{ij}$.

4. The d -connection obtained in Theorem 3.1 will be called *the Cartan connection of the GL-space* (M, a_{ij}) .

From Theorem 3.1 we get

Theorem 3.2. *The Cartan connection $CT = (N_j^i, L_{jk}^i, C_{jk}^i)$ of the GL-space $(M, \bar{g}_{ij})$ where $\bar{g}_{ij} = \theta g_{ij} + (\psi - \theta) \overset{\circ}{\ell}_i \overset{\circ}{\ell}_j$ has the following coefficients:*

$$(3.26) \quad L_{jk}^i = F_{jk}^i + \frac{1}{2\theta} (\theta_j \overset{\circ}{h}_k^i + \theta_k \overset{\circ}{h}_j^i - \theta^i \overset{\circ}{h}_{jk}) - \frac{\theta_0}{2\theta} \overset{\circ}{C}_{jk}^i$$

$$(3.27) \quad C_{jk}^i = \overset{\circ}{C}_{jk}^i + \frac{1}{2\theta} (\dot{\theta}_j \overset{\circ}{h}_k^i + \dot{\theta}_k \overset{\circ}{h}_j^i - \dot{\theta}^i \overset{\circ}{h}_{jk}) \\ + \frac{\psi' F}{\psi} \overset{\circ}{\ell}_j \overset{\circ}{\ell}_k \overset{\circ}{\ell}^i + \frac{\psi - \theta}{F\psi} \left(1 + \frac{\dot{\theta}_0}{2\theta}\right) \overset{\circ}{h}_{jk} \overset{\circ}{\ell}^i$$

$$(3.28) \quad N_j^i = \overset{\circ}{N}_j^i + \frac{\theta_0}{2\theta} \overset{\circ}{h}_j^i.$$

Corollary 3.3. *The coefficients of the Cartan connection for a m -homogeneous Lagrange manifold are given as follows*

$$(3.29) \quad L_{jk}^i = F_{jk}^i, \quad N_j^i = \overset{\circ}{N}_j^i$$

$$(3.30) \quad C_{jk}^i = \overset{\circ}{C}_{jk}^i + \frac{m-2}{2F} (\delta_j^i \overset{\circ}{\ell}_k + \delta_k^i \overset{\circ}{\ell}_j) \\ + \frac{m-2}{(m-1)F} \overset{\circ}{h}_{jk} \overset{\circ}{\ell}^i - \frac{m-2}{2F} \overset{\circ}{\ell}_j \overset{\circ}{\ell}_k \overset{\circ}{\ell}^i$$

(see [4], [12]).

Corollary 3.4. *The Cartan connection of GL -space $(M, \bar{g}_{ij})$, $\bar{g}_{ij} = \theta g_{ij} + (1 - \theta) \overset{\circ}{\ell}_i \overset{\circ}{\ell}_j$ with $\theta = \text{const.}$ are as follows:*

$$(3.31) \quad L_{jk}^i = F_{jk}^i, \quad N_j^i = \overset{\circ}{N}_j^i$$

$$(3.32) \quad C_{jk}^i = \overset{\circ}{C}_{jk}^i + \frac{1 - \theta}{F^2} h_{jk} y^i.$$

Proposition 3.5. *The Cartan connection given in Theorem 3.2 has the property $C_{j0}^i = 0$ iff θ is 0-homogeneous and $\psi = \text{const.}$ ($n \geq 2$).*

PROOF. Transvecting in (3.27) by y^k we get:

$$0 = C_{j0}^i = -\frac{1}{2\theta} \dot{\theta}_0 h_j^i + \frac{\psi' F^2}{\psi} \overset{\circ}{\ell}_j \overset{\circ}{\ell}^i.$$

Transvecting again by y^j we obtain $\psi' = 0$ and then $\dot{\theta}_0 = 0$, therefore θ is 0-homogeneous and $\psi = \text{const.}$

Let us see now about geodesics of GL -space $(M, \bar{g}_{ij})$ with

$$\bar{g}_{ij} = \theta g_{ij} + (\psi - \theta) \overset{\circ}{\ell}_i \overset{\circ}{\ell}_j.$$

The energy functional is given by

$$\bar{\mathcal{E}} = \psi(F^2)F^2.$$

We remark that $(M, \bar{\mathcal{E}})$ is a φ -Lagrange space with $\varphi(t) = \psi(t) \cdot t$ and from a result of [4] the geodesics of the Finsler space $(M, F(x, y))$ parametrized by arc length coincide with the extremals of energy integral and with the paths of $\overset{\circ}{N}_j^i$.

From (3.28) we obtain that $\overset{\circ}{N}_j^i$ and N_j^i have the same paths and from the deflection free property these are the h -paths of GL -space $(M, \bar{g}_{ij})$.

Therefore, we can state:

Theorem 3.6. *For the GL -space $(M, \bar{g}_{ij})$ the geodesic of the associate Finsler space (parametrized by arc length), the extremals of the action integral and the paths of nonlinear connection N_j^i (or $\overset{\circ}{N}_j^i$) coincide. The horizontal lifts of these curves are h -paths.*

Remark. For the metric $\bar{g}_{ij}(x, y) = \gamma_{ij}(x) + \beta(x, y)\overset{\circ}{\ell}_i\overset{\circ}{\ell}_j$ where $\gamma_{ij}(x)$ is a Riemannian metric used in the geometric relativistic optic (see [17]) we can make a conformal change $a_{ij} = e^{-\ell n(1+\beta(x, y))}(\gamma_{ij} + \beta(x, y)\overset{\circ}{\ell}_i\overset{\circ}{\ell}_j)$ and the new GL -space (M, a_{ij}) have the properties of these from Theorem 3.2 and Theorem 3.6.

Acknowledgements. I would like to express my sincere gratitude to Professor P.L. ANTONELLI for many stimulating discussions on this paper and for his support during my visit at the University of Alberta. I would also like to thank VIVIAN SPAK for her excellent typesetting of this manuscript.

References

- [1] L. M. ABATANGELO, S. DRAGOMIR and S. HÖJÖ, On submanifolds of Finsler spaces, *Tensor, N.S.* **47** (1988), 272–285.
- [2] M. ANASTASIEI and P. L. ANTONELLI, The differential geometry of Lagrangians which generate sprays, *Lagrange and Finsler Geometry, Kluwer Academic Publishers*, 1996, 15–34.
- [3] P. L. ANTONELLI, R. BRADBURY, V. KRIVAN and H. SHIMADA, A dynamical theory of heterocrony: Time-sequencing changes in ecology, development and evolution, *Jour. of Biol. Syst.* **1** (1993), 451–487.
- [4] P. L. ANTONELLI and D. HRIMIUC, A new class of spray generating Lagrangians, *Lagrange on Finsler Geometry, Kluwer Academic Publishers*, 1996, 81–92.
- [5] P. L. ANTONELLI and D. HRIMIUC, On the theory of φ -Lagrange manifolds with applications in Biology and Physics, *Nonlin. World* **3** (1996), 299–333.
- [6] P. L. ANTONELLI, R. S. INGARDEN and M. MATSUMOTO, The Theory of Sprays and Finsler Space with Applications in Physics and Biology, *Kluwer Academic Press, Dordrecht*, 1993, p. 350.
- [7] A. K. ARINGAZIN and G. S. ASANOV, Problems of Finslerian theory of gauge fields and gravitation, *Rep. Math. Phys.* **25** (1988), 183–249.
- [8] G. S. ASANOV and T. KAWAGUCHI, A post Newtonian estimation for the metric $\gamma_{ij} + \alpha c^{-2}y_i y_j$, *Tensor N. S.* **49** (1990), 99–102.
- [9] R. G. BEIL, New class of Finsler metrics, *Int. J. Theor. Phys.* **28** (1989), 659–667.
- [10] J. EHLERS F. F. E. PIRANI and A. SCHILD, In General Relativity (Raifeartaigh, L.O., ed.), *Oxford*, 1972.
- [11] M. HASHIGUCHI, On conformal transformation of Finsler spaces, *J. Math. Kyoto Univ.* **16-1** (1976), 25–50.
- [12] S. HÖJÖ, Structures of fundamental function of S_3 -like Finsler spaces, *J. Math. Kyoto Univ.* **21** (1981), 787–807.
- [13] S. HÖJÖ, Fundamental metric functions and generalized connections on Finsler manifold, Proc. Romainian-Japanese, *Coloq. on Finsler Geometry, Brasov*, 1984, 67–84.

- [14] H. ISHIKAWA H., On generalized Finsler metric $LL_{ij} - \alpha L_i L_j$, *Rep. Math. Phys.* **25** (1988), 243–247.
- [15] T. KAWAGUCHI and R. MIRON, A Lagrangian model for gravitation and electromagnetism, *Tensor N. S.* **48** (1989), 153–168.
- [16] M. MATSUMOTO, Foundations of Finsler Geometry and Special Finsler Spaces, *Kaiseisha Press, Otsu, Japan*, 1986.
- [17] R. MIRON, Metrical Finsler structures and metrical Finsler connections, *J. Math Kyoto Univ.* **23** (1983), 219–224.
- [18] R. MIRON and M. ANASTASIEI, Lagrange spaces, Theory and Applications, *Kluwer Academic Press, Dordrecht*, 1994, p. 285.
- [19] R. MIRON and R. TAVAKOL, Geometry of space-time and generalized Lagrange spaces, *Publ. Math. Debrecan* **44**/1-2 (1994), 167–174.
- [20] R. MIRON and T. KAWAGUCHI, Relativistic geometrical optics, *Int. J. Theor. Phys.* **30** (1991), 1521–1543.
- [21] I. W. ROXBURGH, Post-Newtonian constraints on the Lagrange metric $\gamma_{ij}(x) + \alpha y_i y_j$, *Tensor N. S.* **51** (1992), 59–63.
- [22] H. SHIMADA, Cartan-like connections of special generalized Finsler spaces, Proc. Conf. Diff. Geom. and Applications Brno, Czechoslovakia, *World Scientific, Singapore*, 1990, 270–275.
- [23] S. WATANABE S. IKEDA and F. IKEDA, On metrical Finsler connection of a generalized Finsler metric $g_{ij} = e^{2\sigma(x,y)} \gamma_{ij}(x)$, *Tensor, N. S.* **40** (1983), 97–102.

D. HRIMIUC
 DEPARTMENT OF MATHEMATICAL SCIENCES
 UNIVERSITY OF ALBERTA
 EDMONTON, ALBERTA
 T6G 2G1
 CANADA

(Received April 11, 1997)