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On associative algebras which are sum of two almost commutative subalgebras

By A. PETRAVCHUK (Kiev)

Abstract. The following theorem is proved: if an associative algebra A over an arbitrary field can be decomposed into a sum $A = B + C$ with almost commutative subalgebras B and C (an algebra is called here almost commutative if it has a commutative ideal of finite codimension) then the algebra A possesses a nilpotent ideal I such that the quotient algebra A/I is almost commutative.

1. Introduction

In the paper of K.I. Beidar and A.V. Mikhalev [4] the following problem was stated: whether a sum $R = A + B$ of two associative PI-rings A and B is a PI-ring? There are positive answers to this question for some classes of rings A and B which are near to commutative [4], [5] (every sum of two commutative rings is a PI-ring [2]).

Any associative algebra over an arbitrary field which has a commutative ideal of finite codimension (we will call a such algebra almost commutative) is a PI-algebra and the question about the structure of a sum of two such algebras is of interest. In this paper, the following result is obtained: every sum of two almost commutative algebras contains a nilpotent ideal such that the quotient algebra on this ideal is almost commutative, in particular, every such sum is a PI-algebra. Similar question in group theory i.e. the question about structure of the product of two almost abelian

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(finite-by-abelian) groups is open although it was proved in some cases that this product is almost soluble (see [7], [6] and others).

All considered algebras and rings are associative, the ground field F is arbitrary. The centre of an algebra (or a ring) A is denoted by $Z(A)$. For F-subspaces X and Y of an algebra A, as usual, $[X,Y]=\{xy-yx \mid x\in X, y\in Y\}$; for a subset S of A and for a subalgebra B of A we will denote by $\mathrm{Ann}^l_B(S)$ and $\mathrm{Ann}^r_B(S)$ the left and corresponding the right annulator of S in the subalgebra B.

The following statement is the main result of this paper:

Theorem. *Let A be an associative algebra over an arbitrary field which is decomposable into a sum $A=B+C$, where B and C are almost commutative subalgebras of A. Then the algebra A contains a nilpotent ideal I such that the quotient algebra A/I is almost commutative.*

Previously, we prove a series of lemmas, some these results can be of interest out of this work. In particular, the Proposition 2 which is used in the proof of the Theorem is an extension (for algebras over a field) of the result of O.H. Kegel about sum of two nilpotent associative rings [8].

Lemma 1 (See for example [9]). *Let A be an associative algebra over an arbitrary field and B a subalgebra of A with $\dim A/B<\infty$. Then B contains an ideal I of algebra A such that $\dim A/I<\infty$.*

Lemma 2. *Let I be an one-sided or two-sided commutative ideal of a ring R. Then R has an ideal J such that $J^2=0$ and $(I+J)/J\subseteq Z(R/J)$.*

Proof. Let I be for example a right ideal of the ring R and $i, i_1\in I$, $r\in R$ any elements. Then

$$i_1(ir-ri)=i(i_1r)-(i_1r)i=0$$

because $i_1r\in I$ and $[I,I]=0$. Hence $I[I,R]=0$. Let $T=\mathrm{Ann}^r_R(I)$. Clearly, T is an ideal of the ring R and $[I,R]\subseteq T$. For any element $t\in T$ it holds $(ir-ri)t=irt=0$ (because $rt\in T$, $IT=0$) and therefore $[I,R]T=0$. Now let $J=\mathrm{Ann}^l_T(T)$. It is obvious that J is an ideal of the ring R and $J^2=0$. As $[I,R]\subseteq J$, we have $(I+J)/J\subseteq Z(R/J)$. The case of the left ideal can be considered analogously.

Lemma 3. *Let A be an associative algebra and I an ideal of the algebra A. If J is an ideal of the subalgebra I then it holds:*

(1) if subalgebra J is nilpotent then J lies in a nilpotent ideal J_I of the algebra A and $J_I \subseteq I$;

(2) if subalgebra J is finite dimensional then J lies in an ideal J_I of the algebra A such that $J_I \subseteq I$ and J_I contains a nilpotent ideal T of the algebra A with $\dim J_I/T < \infty$.

Proof. (1) See for example [1, Lemma 1.1.5].

(2) Let J_I be the smallest ideal of the algebra A which contains J and lies in the ideal I of A. Since $J_I^3 \subseteq J$ (see [1, Lemma 1.1.5]), J_I^3 is a finite dimensional ideal of the algebra A. If $J_I^3 = 0$ then we set $T = J_I$ and the statement (2) is proved. Let $J_I^3 \neq 0$ and $T = \mathrm{Ann}^l_{J_I}(J_I^3)$. Clearly, T is an ideal of the algebra A and $(T \cap J_I^3)^2 = 0$. Further, $T/(T \cap J_I^3) \simeq T + J_I^3/J_I^3$ is a nilpotent algebra as a subalgebra of the nilpotent algebra J_I/J_I^3 and therefore the ideal T is nilpotent. Since $\dim J_I^3 < \infty$, we have, clearly, $\dim J_I/T < \infty$. The statement (2) and the Lemma are proved.

Lemma 4. *Let R be an associative ring and I any commutative ideal of R. If the quotient ring R/I is commutative or nilpotent then the ring R contains some nilpotent ideal with the commutative quotient ring.*

Proof. We may restrict ourselves by Lemma 2 only to the case $I \subseteq Z(R)$. First, let the quotient ring R/I be commutative. For any elements $i \in I$, $r_1, r_2 \in R$ we have

$$ir_1r_2 - r_2ir_1 = 0 = i[r_1, r_2]$$

(because $ir_1 \in I \subseteq Z(R)$) and therefore $I[R, R] = 0$. Let J denote the annulator of the ideal I in I. Then J is an ideal of the ring R with $J^2 = 0$ and $[R, R] \subseteq J$ (because $[R, R] \subseteq I$). Thus the quotient ring R/J is commutative and and the proof is complete in the case of commutative quotient ring R/I. Now let the quotient ring R/I be nilpotent. If $(R/I)^2$=0 then this case follows from the above considered case. Let the statement of Lemma be true for an arbitrary ring R with $(R/I)^n = 0$, $n \geq 2$, prove it for a ring R with condition $(R/I)^{n+1} = 0$. Denote $N = R^2 + I$. Clearly, N/I is an ideal of the quotient ring R/I and $(N/I)^n = 0$. By inductive assumption the subring N contains some nilpotent ideal T such that the quotient ring N/T is commutative. By Lemma 3 T lies in some nilpotent ideal S of the ring R with $S \subseteq N$. Then the quotient ring $\overline{R} = R/S$ contains a commutative ideal $\overline{N} = N/S$ such that $(\overline{R}/\overline{N})^2 = 0$. As was proved above the ring $\overline{R}$ contains some nilpotent ideal $\overline{J} = J/S$ such that $\overline{R}/\overline{J}$ is commutative. It is obvious that J is a nilpotent ideal of the ring R and the quotient ring R/J is commutative. The proof is complete.

Lemma 5. *Let A be an almost commutative associative algebra and I a commutative ideal of A with $\dim A/I < \infty$. Then:*

(1) $[A, A]I$ lies in some nilpotent ideal of the algebra A;

(2) for some nilpotent ideal J the quotient algebra A/J contains a finite dimensional ideal T/J such that the quotient algebra A/T is commutative.

Proof. (1) If $I \subseteq Z(A)$ then we have for any elements $a_1, a_2 \in A$ and $i \in I$

$$(a_1 a_2 - a_2 a_1)i = (a_1 i)a_2 - a_2(a_1 i) = 0,$$

because $a_1 i \in I \subseteq Z(A)$ and therefore $[A, A]I = 0$. Now if $I \nsubseteq Z(A)$ then going to the quotient algebra A/J on some nilpotent ideal J with $I \subseteq Z(A/J)$ (it exists by Lemma 2) we get $[A, A]I \subseteq J$.

(2) We can assume, without loss of generality, by Lemma 2 that $I \subseteq Z(R)$. Clearly, $T = \mathrm{Ann}_A(I)$ is an ideal of the algebra A and by part 1 of this Lemma $T \supseteq [A, A]$. Let denote $J = T \cap I$. Obviously, $J^2 = 0$ and T/J is a finite dimensional ideal of the algebra A/J. At that the quotient algebra $(A/J)/(T/J) \simeq A/T$ is commutative.

For convenience and shortness we introduce the following:

Definition 1. An associative algebra A over an arbitrary field will be called an *NCF*-algebra if it contains a nilpotent ideal with almost commutative quotient algebra.

An ideal of an associative algebra will be called an *NCF*-ideal if it is an *NCF*-algebra.

In particular, every nilpotent, commutative and finite dimensional algebras are *NCF*-algebras by this definition.

Proposition 1. *The following statements hold:*

(1) every subalgebra and every quotient algebra of an NCF-algebra are NCF-algebras;

(2) if A and B are NCF-algebras then the direct product $A \times B$ is an NCF-algebra;

(3) every extension of an NCF-algebra by other NCF-algebra is an NCF-algebra.

PROOF. The statements (1) and (2) of the Proposition are obvious. Prove the statement (3), i.e. show that an algebra A is an *NCF*-algebra if it contains an *NCF*-ideal B such that A/B is also an *NCF*-algebra. Consider some cases previously:

(a) The quotient algebra A/B is finite dimensional.

Let I be a nilpotent ideal of the *NCF*-algebra B such that B/I is almost commutative. By part 1 of Lemma 3 I lies in some nilpotent ideal of the algebra A and the latest lies in B. Then we can assume, without loss of generality, $I = 0$ i.e. B is commutative. Since $\dim A/B < \infty$, the algebra A is almost commutative by Lemma 1 and therefore it is an *NCF*-algebra.

(b) The ideal B is finite dimensional.

The right annulator $C = \mathrm{Ann}^r_A(B)$ is an ideal of the algebra A, and since $C/(B \cap C) \simeq C + B/B$ is an *NCF*-algebra, the ideal C is an *NCF*-algebra in view of equality $(B \cap C)^2 = 0$. It follows from the inequality $\dim A/C < \infty$ and part (a) of this proof that A is an *NCF*-algebra.

(c) The ideal B is commutative.

In the *NCF*-algebra A/B there exists a nilpotent ideal N/B such that quotient algebra $(A/B)/(N/B) \simeq A/N$ is almost commutative. Without loss of generality one can assume in view of Lemma 4 and part 1 of Lemma 3 that the ideal N is commutative. Denote by S/N any commutative ideal of finite codimension in algebra A/N. By Lemma 4 and part 1 of Lemma 3 we can assume also S is commutative. Since $\dim A/S < \infty$, we obtain that A is an *NCF*-algebra.

Now prove the statement (3) in general case. Let N be any nilpotent ideal of the subalgebra B such that the quotient algebra B/N is almost commutative. By part 1 of Lemma 3 one can assume without loss of generality $N = 0$ i.e. the ideal B is almost commutative. Analogously, by part 2 of Lemma 5 and part 1 of Lemma 3 we can consider the subalgebra B has a finite dimensional ideal T with commutative quotient algebra B/T. Similarly, one can assume the subalgebra T of A lies in some finite dimensional ideal T_B of algebra A such that $T_B \subseteq B$ and B/T_B is commutative. Then A/T_B is an *NCF*-algebra in view of part (c) of this proof. Since $\dim T_B < \infty$, the algebra A is an *NCF*-algebra by part (b) of this proof. The proof is complete.

It follows from Lemma 1 and Proposition 1 the next statement:

Corollary 1. *If an associative algebra A has an NCF-subalgebra B and $\dim A/B < \infty$ then A is an NCF-algebra.*

Lemma 6 ([2], [3, Th. 2.2])**.** *Let R be an associative ring which is decomposable into a sum $R = A + B$ of two commutative subrings A and B. Then R has an ideal I with $I^2 = 0$ and commutative quotient ring R/I.*

Lemma 7. *Let A be an associative algebra over an arbitrary field F, B and C commutative subalgebras of A and let I be an ideal of A which lies in the F-subspace $B + C$. Then I is an NCF-ideal.*

Proof. Let $I_B = \{b \in B \mid \text{there exists } i \in I \text{ of the form } i = b + c, c \in C\}$ i.e. I_B is a projection of the ideal I into subalgebra B. Analogously, define the projection I_C of I on subalgebra C. Obviously, it holds for elements $i_1, i_2 \in I$, $i_1 = b_1 + c_1$, $i_2 = b_2 + c_2$, where $b_i \in B$, $c_i \in C$, $i = 1, 2$ the equality

$$i_1 i_2 = (b_1 + c_1)(b_2 + c_2) = i_1 c_2 + c_1 i_2 + b_1 b_2 - c_1 c_2.$$

Thus $b_1 b_2 - c_1 c_2 \in I$, and hence I_B, I_C are subalgebras of B and corresponding C. It is easy to see that $I_B + I_C$ is a subalgebra of A, and since the subalgebras I_B and I_C are both commutative, $I_B + I_C$ is an *NCF*-algebra by Lemma 6. Then the ideal I which lies in $I_B + I_C$ is an *NCF*-algebra. The proof is complete.

For convenience we give the following definition:

Definition 2. An associative algebra A over an arbitrary field F decomposable into a sum $A = B + C$ of two almost commutative subalgebras B and C will be called a minimal BM-counter-example if A satisfies the following conditions:

(1) A is not an *NCF*-algebra;

(2) the subalgebras B and C have commutative ideals B_0 and corresponding C_0 such that $\dim B/B_0 + \dim C/C_0 < \infty$ and the number $\dim A/(B_0 + C_0)$ is the smallest;

(3) the algebra A has not nonzero ideals which lie in the F-subspace $B_0 + C_0$ from the condition (2).

Lemma 8. *Let $A = B + C$ be a minimal BM-counter-example. Then for every nonzero ideal I of A the quotient algebra A/I is an NCF-algebra. Besides, the algebra A has not nonzero NCF-ideals.*

Proof. Let $\dim A/(B_0 + C_0) = n$ where B_0 and C_0 are the commutative ideals of subalgebra B and corresponding C from Definition 2. If $n = 0$ then the algebra A is a sum of two commutative subalgebras B_0 and C_0 and hence it is an *NCF*-algebra by Lemma 6. This contradicts to the choice of the algebra A and therefore $n \geq 1$. Let I be a nonzero ideal of the algebra A such that A/I is not an *NCF*-algebra. Denote

$$\overline{A} = A + I/I, \quad \overline{B} = B + I/I, \quad \overline{C} = C + I/I,$$

$$\overline{B_0} = B_0 + I/I, \quad \overline{C_0} = C_0 + I/I.$$

Let $m = \dim \overline{A}/(\overline{B}_0 + \overline{C}_0)$. By the Definition 2 it holds $I \not\subseteq B_0 + C_0$ and hence $m < n$. Denote by $\overline{T}$ the sum of all ideals of the algebra $\overline{A}$ which lie in the F-subspace $\overline{B}_0 + \overline{C}_0$. The ideal $\overline{T}$ is an *NCF*-algebra by Lemma 7 and therefore the quotient algebra $\overline{A}/\overline{T}$ is not an *NCF*-algebra in view of the choice of A and Proposition 1.

Since the F-subspace $(\overline{B}_0 + \overline{C}_0)/\overline{T}$ does not contain nonzero ideals of the algebra $\overline{A}/\overline{T}$ and its codimension in $\overline{A}/\overline{T}$ is equal to m, $m < n$, it contradicts to the choice of A. The obtained contradiction proves that A/I is an *NCF*-algebra.

Now let J be a nonzero *NCF*-ideal of the algebra A. As has just been proved A/J is an *NCF*-algebra, and then A is an *NCF*-algebra by Proposition 1. The latest is impossible and hence A has not nonzero *NCF*-ideals. The proof is complete.

Lemma 9. *Let R be an associative ring which is decomposable into a sum $R = A + B$ of two subrings A and B and let R_0 be a subring of R with $R_0 \supseteq B$. If R_0 contains an ideal A_0 of the subring A then R_0 contains some ideal J of the ring R such that $J \supseteq A_0$.*

Proof. Consider the subring $J = A_0 + BA_0 + A_0B + BA_0B$ of the ring R. Clearly, $J \subseteq R_0$ and $A_0 \subseteq J$. As A_0 is an ideal of the subring A, J obviously, is an ideal of the ring R.

Lemma 10. *Let $A = B+C$ be a minimal BM-counter-example where subalgebras B and C satisfy all conditions of the Definition 2 and let $A_1 = B + C_1$ $(C_1 \subseteq C)$ be a subalgebra of A. If A_1 is not an NCF-algebra then for some NCF-ideal J of subalgebra A_1 the quotient algebra A_1/J is a minimal BM-counter-example.*

Proof. Let $B_0 \subseteq B$ and $C_0 \subseteq C$ be commutative ideals of the sublagebra B and corresponding C from the Definition 2 and let $n = \dim A/(B_0 + C_0)$. Denote by C_1' the subalgebra in C which is generated by C_1 and $C \cap B$. Clearly,

$$B + C_1 = B + C_1', \qquad C_1' \cap B = C \cap B,$$

and therefore one can assume, without loss of generality, that $C_1' = C_1$ and hence $C_1 \cap B = C \cap B$. It follows from the latest equality that $C_0 \cap A_1 = C_0 \cap C_1$. Indeed, let $x \in C_0 \cap A_1$. Then $x = c_0, c_0 \in C_0, x = b + c_1$ for some $c_1 \in C_1$, $b \in B$. Hence $b = c_0 - c_1 \in B \cap C = B \cap C_1$ and $x \in C_1$. But then $x \in C_0 \cap C_1$ and $C \cap A_1 \subseteq C_0 \cap C_1$, because the element x has been chosen in any way. The inclusion $C_0 \cap C_1 \subseteq C_0 \cap A_1$ is obvious.

Since $A_1 \cap (B_0 + C_0) \supseteq B_0$, it holds $A_1 \cap (B_0 + C_0) = B_0 + (C_0 \cap A_1)$ and therefore as proved above

$$A_1 \cap (B_0 + C_0) = B_0 + (C_0 \cap C_1).$$

Now denote by J the sum of all ideals of the algebra A_1 which lie in $B_0 + (C_0 \cap C_1)$. By Lemma 7 J is an *NCF*-ideal of the algebra A_1 and A_1/J is not an *NCF*-algebra (because A_1 is not an *NCF*-algebra). It is easy to see that A/J is a minimal BM-counter-example (in particular, $\dim A_1/((B_0 + C_0) \cap A_1) = n$). The proof is complete.

Lemma 11. *If I is an one-sided finite dimensional ideal of an associative algebra A then A has a nilpotent ideal J such that $(I + J)/J$ lies in some finite dimensional (two-sided) ideal of the quotient algebra A/J.*

Proof. Let I be for example a right ideal of the algebra A and $S = \mathrm{Ann}_A^r(I)$. Clearly, S is an ideal of A and $\dim A/S < \infty$. Further, $T = \mathrm{Ann}_A^l(S)$ is also an ideal of A, $T \supseteq I$ and $J = T \cap S$ is an ideal of the algebra A with $J^2 = 0$. It is easy to see that $\dim T/J < \infty$ and $I + J/J \subseteq T/J$. The case of a left ideal of A can be considered analogously. The Lemma is proved.

Lemma 12. *Let A be an associative algebra over an arbitrary field F and a is an element in A such that* $\dim A/\operatorname{Ann}^r_A(a) < \infty$ $(\dim A/\operatorname{Ann}^l_A(a) < \infty)$. *Then the element a belongs to a finite dimensional right (corresponding left) ideal of the algebra A.*

Proof. Denote by C the right annulator $\operatorname{Ann}^r_A(a)$ and let $\dim A/C < \infty$. Choose a complete system of representatives $\{h_1, \dots , h_n\}$ of the congruence classes of A by C. We will show that F-subspace $I = \langle a, ah_1, \dots , ah_n\rangle$ is a right ideal of the algebra A. If g is any element in A then g is of the form $g = c + \sum_{i=1}^n \alpha_i h_i$ where $c \in C$, $\alpha_i \in F$, $i = 1, \dots , n$. Hence

$$ag = a\left(c + \sum_{i=1}^n \alpha_i h_i\right) = \sum_{i=1}^n \alpha_i a h_i \in I,$$

because $ac = 0$. Further, denote $h_i g = c_i + \sum_{j=1}^n \beta_{ij} h_j$ where $c_i \in C$, $\beta_{ij} \in F$, $i, j = 1, \dots , n$. Then we obtain

$$(ah_i)g = a(h_i g) = a\left(c_i + \sum_{j=1}^n \beta_{ij} h_j\right) = \sum_{j=1}^n \beta_{ij} a h_j \in I.$$

Therefore I is a right ideal of the algebra A, $a \in I$ and $\dim I < \infty$. Analogously, one can consider the case $\dim A/\operatorname{Ann}^l_A(a) < \infty$. The proof is complete.

Let A be a nilpotent algebra, $A \neq 0$. The number $n = n(A)$ such that $A^n = 0$, $A^{n-1} \neq 0$ will be called the index of nilpotency of A and denoted by $n(A)$. The index of nilpotency of the zero algebra we assume to be equal 1.

An associative algebra A will be called almost nilpotent if it has a nilpotent ideal of finite codimension. By $\bar{n}(A)$ will be denoted the smallest nilpotency index of all nilpotent ideals of A of finite codimension in A.

Lemma 13. *If I is a right (left) almost nilpotent ideal of an algebra A then A has a nilpotent ideal J such that $I + J/J$ is a finite dimensional right (corresponding left) ideal of A.*

Proof. Let I be, for example, a right ideal. Let B be any nilpotent ideal of the subalgebra I with $\dim I/B < \infty$ such that $n(B) = \bar{n}(I)$. If $\bar{n}(I) = 1$ then $\dim I < \infty$, that is $B = 0$, and Lemma is proved. First consider the case $\dim I/\operatorname{Ann}_I^r(g) < \infty$ for every element $g \in I$. Let the statement of Lemma be true for algebras with $\bar{n}(I) < k$, prove it for algebras with $\bar{n}(I) = k$. Choose a complete system of representatives $\{g_1, \dots, g_m\}$ of the congruence classes of I by B. Using Lemma 12 one can easy show that there exists a finite dimensional right ideal N of the algebra I with $\{g_1, \dots, g_m\} \subseteq N$. Obviously, $I = B + N$. Then $T = \operatorname{Ann}_B^r(N)$ is a nilpotent ideal of the subalgebra I and $\dim I/T < \infty$. Analogously, $I_0 = \operatorname{Ann}_I^r(I)$ is a nilpotent right ideal of the algebra A and $I_0 \supseteq T \cap B^{k-1}$. Then as is well known (see for example [1, Lemma 1.1.2]) I_0 lies in some nilpotent ideal S of the algebra A. At that the quotient algebra $\overline{A} = A/S$ has the right almost nilpotent ideal $\overline{I} = I + S/S$. Since $T \cap B^{k-1} \subseteq S$ then the ideal $\overline{T} = T + S/S$ of the subalgebra $\overline{I}$ has the nilpotency index $\leq k-1$ and therefore $\bar{n}(\overline{I}) \leq k-1$. By inductive assumption there exists in $\overline{A}$ some nilpotent ideal $\overline{J} = J/S$ such that $\overline{I} + \overline{J}/\overline{J}$ is a finite dimensional right ideal of the algebra $\overline{A}/\overline{J}$. Then J is a nilpotent ideal of the algebra A and $I + J/J$ is a finite dimensional right ideal of the quotient algebra A/J.

Now let $I_1 = \{i \in I \mid \dim I/\operatorname{Ann}_I^r(i) < \infty\}$. It is easy to see that I_1 is a right ideal of the algebra A, $I_1 \cap B$ is a nilpotent ideal in I_1 and $\dim I_1/(I_1 \cap B) < \infty$. Besides, $B^{k-1} \subseteq I_1$, because $B^{k-1}B = 0$ and $\dim I/B < \infty$. As has just been proved there exists in A a nilpotent ideal U such that $I_1 + U/U$ is a finite dimensional right ideal of the algebra A/U. One can assume without loss of generality (by Lemma 11) that the algebra A has a finite dimensional ideal M such that $M \supseteq B^{k-1}$. By inductive assumption (induction on $\overline{n}(I)$)the quotient algebra A/M contains a nilpotent ideal V/M such that $(I + V/M)/(V/M)$ is a finite dimensional right ideal of the algebra $(A/M)/(V/M)$. Let $V_1 = \operatorname{Ann}_V^r(M)$. It is easy to see that V_1 is a nilpotent ideal of the algebra A and $I + V_1/V_1$ is a finite dimensional right ideal of the algebra A/V_1.

The case of the left ideal I can be considered analogously. The Lemma is proved.

The statements below follow from Lemmas 11 and 13.

Corollary 2. *Let A be an associative algebra and I a right (left) almost nilpotent ideal of the algebra A. Then I lies in some almost nilpotent ideal of the algebra A.*

Corollary 3. *If an associative algebra A has an almost nilpotent ideal I with almost nilpotent quotient algebra A/I then the algebra A is almost nilpotent.*

Proof. One can assume, without loss of generality, that $\dim I < \infty$ (in view of Lemma 3). Let J/I be a nilpotent ideal of the quotient algebra A/I such that $\dim A/J < \infty$ and let $C = \mathrm{Ann}^r_J(I)$. Obviously, C is an ideal of the algebra A and $\dim J/C < \infty$ (and hence $\dim A/C < \infty$). Since $(C \cap I)^2 = 0$ and $C/(C \cap I) \simeq C + I/I$ is a nilpotent algebra then the ideal C is nilpotent. The proof is complete.

Proposition 2. *If an associative algebra A over an arbitrary field is decomposable into a sum $A = B + C$ with almost nilpotent subalgebras B and C of A then the algebra A is almost nilpotent.*

Proof. Let the statement of the Proposition be false. Choose among all counter-examples to the Proposition an algebra $A = B + C$ with the smallest sum $\bar{n}(B) + \bar{n}(C)$. Clearly, $\bar{n}(B) \geq 2$ and $\bar{n}(C) \geq 2$ (if, for example, $\bar{n}(B) = 1$ then $\dim B < \infty$ and therefore the algebra A is almost nilpotent in contradiction to the our assumption). Denote by B_0 and C_0 some nilpotent ideals of the subalgebra B and corresponding of the subalgebra C such that $\dim B/B_0 + \dim C/C_0 < \infty$ and $n(B_0) = \bar{n}(B)$, $n(C_0) = \bar{n}(C)$. Let $I = B_0^{\bar{n}(B)-1}$. It is easy to see that $IB_0 = B_0 I = 0$ and $A_0 = B + IC$ is a subalgebra from A of the form $A_0 = B + C_1$ where $C_1 = C \cap A_0$. Note that B_0 is a right nilpotent ideal of the subalgebra A_0. Then the right ideal B_0 lies as is well known in some (two-sided) nilpotent ideal S of the subalgebra A_0. The almost nilpotent subalgebra $C_1 + S/S$ is of finite codimension in A_0/S and by Lemma 1 the quotient algebra A_0/S is almost nilpotent. Then the algebra A_0 is almost nilpotent.

It is eeasy to see that $I + IC$ is a right ideal of the algebra A, and since $I + IC \subseteq A_0$, the subalgebra $I + IC$ is almost nilpotent. Further, $I + IC$ lies by Corollary 2 in some almost nilpotent ideal T of the algebra A. The quotient algebra $\overline{A} = A/T$ is decomposable into a sum $\overline{A} = \overline{B} + \overline{C}$ where $\overline{B} = B + T/T, \overline{C} = C + T/T$. Since $I \subseteq T$, we have $\bar{n}(\overline{B}) < \bar{n}(B)$ and therefore the quotient algebra A/T is almost nilpotent by choice of the algebra A. In view of Corollary 3 the algebra A is almost nilpotent. It contradicts to the choice of A and the proof is complete.

Lemma 14. *Let A be an associative algebra without nonzero NCF-ideals decomposable into a sum $A = B + C$ of subalgebras B and C which contain commutative ideals $B_0 \subseteq B$ and $C_0 \subseteq C$ such that $\dim B/B_0 + \dim C/C_0 < \infty$. If $A_1 = B + B_0C$ is an NCF-subalgebra of A then the subalgebra $C_1 = C \cap A_1$ is almost nilpotent.*

Proof. Since B_0 is an ideal of subalgebra B, the subspaces B_0C and A_1 are obvius subalgebras of A. Let $g = b + c$ be any element in the F-subspace $B_0C \cap (B_0 + C_0)$ where $b \in B_0, c \in C_0$. It is easy to see that $C_0c = cC_0$ is a two-sided ideal of the algebra C and cC_0 lies in the subalgebra $A_1 = B + B_0C = B + C_1$ Then there exists by Lemma 9 an ideal S of algebra A such that $cC_0 \subseteq S$ and $S \subseteq A_1$. Let A_1 be an *NCF*-subalgebra. Then S is an *NCF*-ideal of the algebra A and by conditions of Lemma $S = 0$. Hence $cC_0 = C_0c = 0$, that is, $c \in \mathrm{Ann}_{C_0}(C_0)$. In view of choice of the element $g = b + c$ this means $(B_0 + B_0C) \cap C_0 \subseteq \mathrm{Ann}_{C_0}(C_0)$. Further, it is easy to see that $\dim C_1/(A_1 \cap C_0) < \infty$ because $\dim C/C_0 < \infty$, and since $\dim B/B_0 < \infty$, we have

$$\dim C_1/((B_0 + B_0C) \cap C_0) < \infty.$$

Obviously, $(\mathrm{Ann}_{C_0}(C_0))^2 = 0$, hence $((B_0 + B_0C) \cap C_0)^2 = 0$ and therefore C_1 is an almost nilpotent subalgebra of the algebra C. The proof is complete.

Lemma 15. *Let $A = B+C$ be a minimal BM-counter-example where subalgebras B and C satisfy conditions of Definition 2. Then both subalgebras B and C are not almost nilpotent.*

Proof. Let the statement of Lemma be false. Then there exist minimal BM-counter-examples of the form $A = B + C$ such that one of the subalgebras B or C is almost nilpotent (by Proposition 2 both subalgebras B and C can not be almost nilpotent simultaneously). Choose among all such counter-examples an algebra $A = B + C$ with almost nilpotent subalgebra B which has the smallest number $n = \bar{n}(B)$. Let B_0' and C_0 be commutative ideals of finite codimension of the algebra B and corresponding C which satisfy conditions of Definition 2. Take a nilpotent ideal N of subalgebra B with $\dim B/N < \infty$ and $n(N) = \bar{n}(B)$ and set $B_0 = B_0' \cap N$. Obviously, B_0 is a commutative nilpotent ideal of finite codimension in B and $n(B_0) = \bar{n}(B)$. It is easy to see that $A_1 = B + B_0C$ is a subalgebra of A. Show that A_1 is an *NCF*-algebra. Denote $I = (B_0)^{\bar{n}(B)-1}$. Then I is a right nilpotent ideal of the subalgebra A_1 and hence I lies in certain nilpotent ideal S of A_1. The quotient algebra A_1/S is decomposable into a sum

$$A_1/S = (B + S)/S + (B_0C + S)/S$$

and $B_0^{\bar{n}(B)-2} + S/S$ is a right nilpotent ideal in A_1/S. Therefore $B_0^{\bar{n}(B)-2} + S/S$ lies in some nilpotent ideal S_1/S of the algebra A_1/S. Repeating this considering one can show that B_0 lies in some nilpotent ideal T of the algebra A_1. Since $\dim B/B_0 < \infty$, the quotient algebra A_1/T is an *NCF*-algebra by Corollary 1 and hence A_1 is an *NCF*-algebra. Obviously, $A_1 = B + C_1$ where $C_1 = C \cap A_1$. The subalgebra C_1 is almost nilpotent (see Lemma 14) and therefore the subalgebra A_1 is almost nilpotent by Proposition 2 as a sum of two almost nilpotent subalgebras B and C_1. Then the right ideal $B_0 + B_0C$ of the algebra A is almost nilpotent and lies by Corollary 2 in some almost nilpotent ideal T_1 of the algebra A. But $T_1 = 0$ by Lemma 8 and hence $B_0 = 0$. It follows from this $\dim A/C < \infty$ and A is an *NCF*-algebra in view of Corollary 1. This contradicts to the choice of algebra A. The proof is complete.

Lemma 16. *Let $A = B + C$ be a minimal BM-counter-example, let B_0 and C_0 be commutative ideals of the subalgebras B and corresponding C from Definition 2. Then $B + B_0C$ and $B + CB_0$ are subalgebras of A and at least one of these subalgebras is not an NCF-algebra.*

Proof. It is easy to see that $A_1 = B + B_0C$ and $A_2 = B + CB_0$ are subalgebras of A because B_0 is an ideal of the subalgebra B. One can immediately check up that B_0C, CB_0 and $A_0 = B + B_0C + CB_0 + CB_0^2C$ are also subalgebras of the algebra A. Suppose the Lemma is false and both subalgebras A_1 and A_2 are *NCF*-algebras. Obviously, it holds $A_1 = B + C_1$, $A_2 = B + C_2$ where $C_1 = C \cap A_1$ and $C_2 = C \cap A_2$. Denote $N_i = C_i \cap \mathrm{Ann}_{C_0}(C_0)$, $i = 1, 2$. Repeating the considerations from the proof of Lemma 14 one can show that $\dim C_i/N_i < \infty$, $i = 1, 2$. Obviously, N_i is an ideal of $C_i, N_i^2 = 0$, $i = 1, 2$. It is easy to see that $A_0 = B + C_3$ where $C_3 = C \cap A_0$. Show that C_3 is almost nilpotent. Since $A_0 = A_1 + A_2 + A_1A_2$, we have $A_0 = B + C_1 + C_2 + C_1C_2$. It follows from this equality that $C_3 = C_1 + C_2 + C_1C_2$. Really, the inclusion $C_1 + C_2 + C_1C_2 \subseteq C_3$ is obvious. Now let $c \in C_3 = C \cap A_0$ be any element. Then $c = b + x_1 + y_1 + \sum_{i=2}^{k} x_iy_i$ where $x_i \in C_1$, $y_i \in C_2$, $i = 1, \ldots, k$.

Then $b \in C$ and hence $b \in C \cap B$. Since $C \cap B \subseteq C_1$ then $c \in C_1 + C_2 + C_1C_2$ and therefore $C_3 \subseteq C_1 + C_2 + C_1C_2$, because the element c was chosen in any way. Thus $C_3 = C_1 + C_2 + C_1C_2$.

Choose a complete system of representatives $\{x_1, \dots , x_m\}$ of the congruence classes of C_1 by N_1 and analogous system $\{y_1, \dots , y_n\}$ of C_2 by N_2. Then

$$N_3 = N_1 + N_2 + \sum_{i=1}^{m} x_i N_2 + \sum_{j=1}^{n} y_j N_1 \subseteq \mathrm{Ann}_{C_0}(C_0),$$

and obviously $\dim C_3/N_3 < \infty$. Since $N_3^2 = 0$, the subalgebra C_3 of A_0 is almost nilpotent.

Show that $A_0 = B + C_3$ is not an *NCF*-subalgebra. Really, let A_0 be conversely an *NCF*-algebra. Then $J = B_0^2 + CB_0^2 + B_0^2C + CB_0^2C$ is an ideal of the algebra A which lies in A_0 and hence J is an *NCF*-ideal. By the conditions of the present Lemma and by Lemma 8 $J = 0$ and hence $B_0^2 = 0$. As a sum of two almost nilpotent subalgebras B and C_3 the subalgebra A_0 is almost nilpotent by Proposition 2. But then $B_0 + B_0C (\subseteq A_0)$ is an almost nilpotent right ideal of the algebra A. By Corollary 2 $B_0 + B_0C$ lies in some almost nilpotent ideal of the algebra A. In view of conditions of this Lemma and Lemma 8 the latest ideal equals zero and hence $B_0 = 0$. Then obviously $\dim A/C < \infty$ and A is an *NCF*-algebra by Corollary 1. This contradicts to the conditions of Lemma and hence $A_0 = B + C_3$ is not an *NCF*-algebra. By Lemma 10 the quotient algebra A_0/J_0 is a minimal BM-counter-example for a some *NCF*-ideal J_0. On other hand $A_0/J_0 = (B + J_0)/J_0 + (C_3 + J_0)/J_0$ where the subalgebra $C_3 + J_0/J_0$ is almost nilpotent and therefore A_0/J_0 is not an BM-counter-example by Lemma 15. The obtained contradiction proves the statement of Lemma.

Proof of the Theorem. Let the statement of the Theorem be false. Choose among all counter-examples to the Theorem a such associative algebra $A = B + C$ over a field F which is not *NCF*-algebra and its F-subspace $B_0 + C_0$ is of the smallest codimension in A where B_0 and C_0 are commutative ideals of subalgebras B and corresponding C. Denote by J_0 the sum of all ideals of the algebra A which lie in $B_0 + C_0$. Then J_0 is an *NCF*-ideal by Lemma 7 and A/J_0 is obviously a BM-counter-example. Thus one can assume, without loss of generality, that $J = 0$ and A is a minimal BM-counter-example.

Note that $A_1 = B + B_0C$ and $A_2 = B + CB_0$ are subalgebras of A, and by Lemma 16 at least one of these subalgebras is not an *NCF*-algebra. Let A_1, for example, be not an *NCF*-algebra. Then the quotient algebra A_1/J_1 for some *NCF*-ideal J_1 of A_1 is a minimal BM-counter-example by Lemma 10. Denote $I_1 = \mathrm{Ann}_{B_0}(B_0)$. Obviously, I_1 is a right nilpotent ideal of the subalgebra $A_1 = B + B_0C$. Then I_1 lies in some nilpotent ideal S_1 of this algebra, and since the quotient algebra A_1/J_1 has not nonzero *NCF*-ideals (see Lemma 8), $S_1 \subseteq J_1$ and hence $I_1 \subseteq J_1$. We have $[B_0, B] \subseteq B_0$ (B_0 is a commutative ideal in B) and repeating the consideration from the proof of Lemma 2 one can show that $[B_0, B] \subseteq \mathrm{Ann}_{B_0}(B_0) = I_1$. But then A_1/J_1 is a sum of almost commutative subalgebra $C_1 + J_1/J_1$ and finite dimensional over its center subalgebra $B + J_1/J_1$. Therefore we can assume, without loss of generality, that in the initial minimal BM-counter-example $A = B + C$ holds the inclusion $B_0 \subseteq Z(B)$ (otherwise we can replace A by A_1/J_1).

Now consider the right ideal $D_0 = \mathrm{Ann}_B(B_0)$ of the subalgebra $A_1 = B + B_0C$ (which is not an *NCF*-algebra by our choice). It is easy to see that $T_0 = D_0 + A_1D_0$ is an ideal of the algebra A_1 and $T_0 \subseteq \mathrm{Ann}^l_{A_1}(B_0)$. Further, denote $I_0 = \mathrm{Ann}^r_{A_1}(T_0)$. Obviously, I_0 is an ideal of the subalgebra A_1, $I_0 \supseteq B_0$ and $(I_0 \cap T_0)^2 = 0$. The quotient algebra $A_1/(I_0 \cap T_0)$ is not an *NCF*-algebra, because in the contrary case the algebra A_1 were also an *NCF*-algebra (in view of nilpotency of the ideal $I_0 \cap T_0$). The latest is impossible. The quotient algebra A_1/I_0 contains an almost commutative subalgebra $C_1 + I_0/I_0$ of finite codimension in A_1/I_0 (because $I_0 \supseteq B_0$) and therefore by Corollary 1 A_1/I_0 is an *NCF*-algebra. Then the quotient algebra $\overline{A}_1 = A_1/T_0$ is not an *NCF*-algebra, because in the contrary case the algebra $A_1/(I_0 \cap T_0)$ were also an *NCF*-algebra in view of embedding $A_1/(I_0 \cap T_0)$ into the product $(A_1/I_0) \times (A_1/T_0)$ and by Proposition 1. Note that $[B, B] \subseteq D_0 \subseteq T_0$. Really, we have for any elements $b_1, b_2 \in B$ and $b_0 \in B_0$

$$(b_1b_2 - b_2b_1)b_0 = b_1b_2b_0 - b_2b_1b_0 = b_1(b_2b_0) - (b_2b_0)b_1 = 0$$

because $b_2b_0 \in B_0 \subseteq Z(B)$. Hence the quotient algebra $\overline{A}_1 = A_1/T_0$ is a sum of the commutative subalgebra $\overline{B} = B + T_0/T_0$ and almost commutative subalgebra $\overline{C}_1 = C_1 + T_0/T_0$ where $C_1 = C \cap A_1$. It easy to see that certain quotient algebra $\overline{A}_1/\overline{S}_1$ is a minimal BM-counter-example for some *NCF*-ideal $\overline{S}_1$ from $\overline{A}_1$. We can assume, without loss of generality, that the subalgebra B in original BM-counter-example $A = B + C$ is commutative. Repeating the above considerations in respect to one of the subalgebras $C + C_0B$ or $C + BC_0$ we can show that there exist minimal BM-counter-examples of the form $A = B + C$ with commutative subalgebras B and C. It is impossible in view of [2] (see Lemma 6). The obtained contradiction proves the Theorem.

Remark. Let $A = B + C$ be the associative algebra from the main theorem and $B_0 \subseteq B$, $C_0 \subseteq C$ be some commutative ideals of B and respectively C of finite codimensions $p = \dim B/B_0$, $q = \dim C/C_0$. By this theorem the algebra A contains a nilpotent ideal I with almost commutative quotient algebra A/I. Let K/I be any commutative ideal of A/I of finite codimension. Then using the main theorem and Lemma 6 one can show that there exist two functions $f(x, y)$ and $g(x, y)$ such that the nilpotency index $n(I) \leq f(p, q)$ and $\dim A/K \leq g(p, q)$.

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A. PETRAVCHUK
DEPARTMENT OF MECHANICS AND MATHEMATICS
KIEV TARAS SHEVCHENKO UNIVERSITY
252033 KIEV VLADIMIRSKAYA STR. 64
UKRAINE

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