

Solutions of linear recursive systems

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PETER R. J. ASVELD [1], [2], furthermore MARJORIE BICKNELL-JOHNSON and GERALD BERGUM [3] investigated sequences determined by initial values and linear recursive systems. The Problem H-351 of the Fibonacci Quarterly, proposed by V. E. HOGGATT, Jr. [4], considers a similar question: Determine the sequences $U = \{U_n\}_{n=1}^{\infty}$, $V = \{V_n\}_{n=1}^{\infty}$ with $U_1 = V_1 = F_1 = F_2 = 1$ and

$$(1) \quad \begin{aligned} U_{n+1} - U_n - V_n - F_{n+1} &= 0, & -U_{n+1} + V_{n+1} - V_n &= 0, \\ F_{n+2} - F_{n+1} - F_n &= 0, & \text{for any } n \geq 1. \end{aligned}$$

The purpose of this paper is to investigate a generalization of these problems.

For a given integer $r \geq 1$ let $X^{(j)} = \{x_n^{(j)}\}_{n=0}^{\infty}$ ($1 \leq j \leq r$) be sequences of real numbers with initial terms $x_0^{(j)}, x_1^{(j)}, \dots, x_{m_j-1}^{(j)}$ ($1 \leq j \leq r$) and let $c_{i,j,t}$ ($1 \leq i, j \leq r$; $0 \leq t \leq m_j$) be fixed real numbers. Suppose that the sequences satisfy the equation system

$$(2) \quad \begin{aligned} &\sum_{t=0}^{m_1} c_{1,1,t} x_{n+t}^{(1)} + \sum_{t=0}^{m_2} c_{1,2,t} x_{n+t}^{(2)} + \cdots + \sum_{t=0}^{m_r} c_{1,r,t} x_{n+t}^{(r)} = 0 \\ &\sum_{t=0}^{m_1} c_{2,1,t} x_{n+t}^{(1)} + \sum_{t=0}^{m_2} c_{2,2,t} x_{n+t}^{(2)} + \cdots + \sum_{t=0}^{m_r} c_{2,r,t} x_{n+t}^{(r)} = 0 \\ &\vdots \\ &\sum_{t=0}^{m_1} c_{r,1,t} x_{n+t}^{(1)} + \sum_{t=0}^{m_2} c_{r,2,t} x_{n+t}^{(2)} + \cdots + \sum_{t=0}^{m_r} c_{r,r,t} x_{n+t}^{(r)} = 0 \end{aligned}$$

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for any natural number n . We assume that

$$(3) \quad \det(c_{i,j,m_j}) \neq 0,$$

where $\det(c_{i,j,m_j})$ is the determinant of the $r \times r$ matrix with entries c_{i,j,m_j} ($0 \leq i, j \leq r$). By (2) and (3) the sequences $X^{(j)}$ are uniquely determined since the initial terms are given. We shall show that these sequences also satisfy a linear recurrence relation.

Let M be the set of operators $A = A(a_0, a_1, \dots, a_m)$, $B = B(b_0, b_1, \dots, b_k), \dots$, defined on the set of sequences of real numbers $X = \{X_n\}_{n=0}^\infty$, $Y = \{Y_n\}_{n=0}^\infty, \dots$, so that

$$\begin{aligned} A(X) = X' &= \{x'_n\}_{n=0}^\infty = \left\{ \sum_{i=0}^m a_i x_{n+i} \right\}_{n=0}^\infty \\ B(X) = X'' &= \{x''_n\}_{n=0}^\infty = \left\{ \sum_{i=0}^k b_i x_{n+i} \right\}_{n=0}^\infty \\ &\vdots \end{aligned}$$

where $a_0, a_1, \dots, a_m; b_0, b_1, \dots, b_k, \dots$ are fixed real numbers. Let $X + Y = \{x_n + y_n\}_{n=0}^\infty$ and $aX = \{ax_n\}_{n=0}^\infty$ for a real number a .

It can easily be checked that

$$(4) \quad A(aX + bY) = aA(X) + bA(Y)$$

for any operator A of M and for any real numbers a and b , that is each element of M is a linear operator. We define the addition and multiplication of operators by

$$(5) \quad (A + B)(X) = A(X) + B(X)$$

and

$$(6) \quad (A \cdot B)(X) = A(B(X))$$

for any sequence X of real numbers.

Let T be the mapping of the set M of operators onto the set $R[x]$ of polynomials with real coefficients defined by

$$T(A) = \sum_{i=0}^m a_i x^i$$

where the operator A is determined by the real numbers $a_0, a_1, \dots, a_m$. The following auxiliary result will be proved at the end of the paper.

Lemma. *The mapping T is an isomorphism between the structures $(M, +, \cdot)$ and $(R[x], +, \cdot)$.*

Using the notation of operators, equation system (2) can be written in the form

$$(7) \quad \begin{aligned} C_{1,1} \left(x^{(1)} \right) + C_{1,2} \left(x^{(2)} \right) + \cdots + C_{1,r} \left(x^{(r)} \right) &= 0^* \\ C_{2,1} \left(x^{(1)} \right) + C_{2,2} \left(x^{(2)} \right) + \cdots + C_{2,r} \left(x^{(r)} \right) &= 0^* \\ \vdots \\ C_{r,1} \left(x^{(1)} \right) + C_{r,2} \left(x^{(2)} \right) + \cdots + C_{r,r} \left(x^{(r)} \right) &= 0^* \end{aligned}$$

where $C_{i,j} \in M$ ($1 \leq i, j \leq r$) is an operator determined by the constants $c_{i,j,1}, c_{i,j,2}, \dots, c_{i,j,m_j}$ and 0^* is the sequence of zeros.

Let Z be the zero element of the ring $(M, +, \cdot)$, i.e. $Z(X) = 0^*$ for any sequence X (Z is determined by zeros).

Using the above notation our main result is as follows:

Theorem. *Let $X^{(j)}$ ($1 \leq j \leq r$) be sequences determined uniquely by their initial terms and by (2) and (3). Then these sequences satisfy the recursive relation*

$$(8) \quad (\det(C_{i,k})) \left(X^{(j)} \right) = 0^* \quad (1 \leq j \leq r),$$

furthermore

$$(9) \quad (\det(C_{i,k})) \neq Z,$$

where $\det(C_{i,k}) \in M$ is the determinant of the $r \times r$ matrix with entries $C_{i,k} \in M$, ($1 \leq i, k \leq r$).

Before proving the Theorem we show some consequences and applications of our result.

If $A = A(a_0, \dots, a_m)$ is an operator and $A(X) = 0^*$ for a sequence X of real numbers, then X is a linear recursive sequence of order m since

$$a_m X_{n+m} + a_{m-1} X_{n+m-1} + \cdots + a_0 X_n = 0$$

for any $n \geq 0$, furthermore the characteristic polynomial of this sequence is

$$a_m x^m + a_{m-1} x^{m-1} + \cdots + a_1 x + a_0 = T(A),$$

where T is the isomorphism defined in the Lemma. So, as a consequence of our theorem we have

Corollary. *Let $X^{(j)}$ ($1 \leq j \leq r$) be sequences such as in the Theorem. Then these are linear recursive sequences of order maximum*

$$m = m_1 + m_2 + \cdots + m_r$$

and their common characteristic polynomial is

$$T(\det(C_{i,j})) = \det(T(C_{i,j})).$$

As an application of the theorem we show the way of solving the system (1). From $U_1 = V_1 = F_1 = F_2 = 1$, by (1), the initial terms of the sequences are $U_0 = V_0 = F_0 = 0$, $U_1 = 1$, $U_2 = 3$, $U_3 = 9$; $V_1 = 1$, $V_2 = 4$, $V_3 = 13$ and $F_1 = F_2 = 1$, $F_3 = 2$. In our case $r = 3$, $X^{(1)} = U$, $X^{(2)} = V$, $X^{(3)} = F$; $m_1 = m_2 = 1$, $m_3 = 2$ and

$$\det(c_{i,j,m_j}) = \begin{vmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0$$

The operators $C_{i,j}$ ($1 \leq i, j \leq 3$) are $C_{1,1}(-1, 1)$, $C_{1,2}(-1, 0)$, $C_{1,3}(0, -1, 0)$, $C_{2,1}(0, -1)$, $C_{2,2}(-1, -1)$, $C_{2,3}(0, 0, 0) = C_{3,1}(0, 0) = C_{3,2}(0, 0) = 0^*$, $C_{3,3}(-1, -1, 1)$ and so, by the Corollary, U, V, F are linear recursive sequences of order maximum 4 with characteristic polynomial

$$\begin{aligned} f(x) &= \det(T(C_{i,j})) = \begin{vmatrix} x-1 & -1 & -1 \\ -x & x-1 & 0 \\ 0 & 0 & x^2-x-1 \end{vmatrix} = \\ &= (x^2 - 3x + 1) \cdot (x^2 - x - 1) \end{aligned}$$

The roots of $f(x)$ are

$$\alpha_1 = \frac{1 + \sqrt{5}}{2}, \alpha_2 = \frac{1 - \sqrt{5}}{2}, \alpha_3 = \frac{3 + \sqrt{5}}{2}, \text{ and } \alpha_4 = \frac{3 - \sqrt{5}}{2}$$

and so, as it is well-known, the terms of the sequences can be expressed as

$$x_n^{(j)} = a_j \alpha_1^n + b_j \alpha_2^n + c_j \alpha_3^n + d_j \alpha_4^n \quad (j = 1, 2, 3)$$

where a_j , b_j , c_j , d_j are fixed real numbers, depending on the initial terms, and they can be calculated by solving a linear equation system generated for $n = 0, 1, 2$ and 3. This way for the sequence V we get

$$a_2 = -\frac{5 + 2\sqrt{5}}{10}, \quad b_2 = \frac{2\sqrt{5} - 5}{10}, \quad c_2 = \frac{2\sqrt{5} + 5}{10}, \quad d_2 = \frac{5 - 2\sqrt{5}}{10}$$

and for the sequence F we obtain

$$a_3 = \frac{\sqrt{5}}{5}, \quad b_3 = -\frac{\sqrt{5}}{5}, \quad c_3 = d_3 = 0.$$

Thus F satisfies also a second order linear recursive relation with characteristic polynomial

$$(x - \alpha_1)(x - \alpha_2) = x^2 - x - 1,$$

hence F is really the Fibonacci sequence.

Another example shows a common generalization of the problems investigated in [1], [2] and [3].

Let $X^{(1)}$ be a sequence of real numbers defined by the initial terms $X_0^{(1)}, X_1^{(1)}, \dots, X_{m-1}^{(1)}$ and by the formula

$$(10) \quad \sum_{i=0}^m a_i x_{n+i}^{(1)} + \sum_{i=1}^k q_i(n) \alpha_i^n = 0 \quad (n \geq 0),$$

where $a_0, a_1, \dots, a_m$ ($a_m \neq 0$) and $\alpha_1, \alpha_2, \dots, \alpha_k$ are fixed real numbers, and $q_i(x)$ are given polynomials with real coefficients of degree $(r_i - 1) \geq 0$ for $i = 1, 2, \dots, k$. It is known that the sequence

$$X^{(2)} = \left\{ x_n^{(2)} \right\}_{n=0}^{\infty} = \left\{ \sum_{i=1}^k q_i(n) \alpha_i^n \right\}_{n=0}^{\infty}$$

is a linear recursive sequence of order $r_1 + r_2 + \dots + r_k$ with characteristic polynomial

$$q(x) = \prod_{i=1}^k (x - \alpha_i)^{r_i}.$$

So there is an operator $B \in M$ such that $B(X^{(2)}) = 0^*$ and $T(B) = q(x)$. Let $A = A(a_0, a_1, \dots, a_m)$ be an operator of M and let E and Z be the unit operator, i.e. $E = E(1)$, and the zero operator, respectively. Then (10) can be written in the form

$$\begin{aligned} A(X^{(1)}) + E(X^{(2)}) &= 0^* \\ Z(X^{(1)}) + B(X^{(2)}) &= 0^* \end{aligned}$$

From this, by the Theorem and the Corollary, it follows that $X^{(1)}$ is a linear recursive sequence with characteristic polynomial

$$T \left(\begin{vmatrix} A & E \\ Z & B \end{vmatrix} \right) = T(A \cdot B) = T(A) \cdot T(B) = \left(\sum_{i=0}^m a_i x^i \right) \cdot \prod_{j=1}^k (x - \alpha_j)^{r_j}$$

Now we prove the Lemma and the Theorem.

PROOF OF THE LEMMA. Let X be a sequence of real numbers and let $A = A(a_0, a_1, \dots, a_m)$ and $B = B(b_0, b_1, \dots, b_k)$ be operators of the set M . Then

$$A(X) = \left\{ \sum_{i=0}^m a_i x_{n+i} \right\}_{n=0}^{\infty} \quad \text{and} \quad B(X) = \left\{ \sum_{i=0}^k b_i x_{n+i} \right\}_{n=0}^{\infty}$$

We can suppose that $m \geq k$ and $b_i = 0$ if $k < i \leq m$. By (5) and (6) we get

$$\begin{aligned} (A+B)(X) &= \left\{ \sum_{i=0}^m a_i x_{n+i} \right\}_{n=0}^{\infty} + \left\{ \sum_{i=0}^k b_i x_{n+i} \right\}_{n=0}^{\infty} = \\ &= \left\{ \sum_{i=0}^m (a_i + b_i) x_{n+i} \right\}_{n=0}^{\infty} \end{aligned}$$

and

$$\begin{aligned} (A \cdot B)(X) &= A \left(\left\{ \sum_{t=0}^k b_t x_{n+t} \right\}_{n=0}^{\infty} \right) = \left\{ \sum_{j=0}^m a_j \sum_{t=0}^k b_t x_{(n+t)+j} \right\}_{n=0}^{\infty} = \\ &= \left\{ \sum_{i=0}^{m+k} \sum_{j+t=i} a_j \cdot b_t \cdot x_{n+i} \right\}_{n=0}^{\infty} \end{aligned}$$

Combining the above equations with the definition of the mapping T , we obtain

$$T(A+B) = \sum_{i=0}^m (a_i + b_i) x^i = \sum_{i=0}^m a_i x^i + \sum_{i=0}^k b_i x^i = T(A) + T(B)$$

and

$$T(A \cdot B) = \sum_{i=0}^{m+k} \sum_{j+t=i} a_j b_t x^i = \left(\sum_{j=0}^m a_j x^j \right) \cdot \left(\sum_{t=0}^k b_t x^t \right) = T(A) \cdot T(B)$$

follow which proves the Lemma since T is obviously a bijective mapping.

PROOF OF THE THEOREM. The Lemma implies that $(M, +, \cdot)$ is an Euclidean ring and the usual properties of determinants are valid if the entries are operators of M .

Let $A^{i,j}$ be the determinant of the $(r-1) \times (r-1)$ matrix that we get from $C_{i,k}$ ($1 \leq i, k \leq r$) by omitting the i^{th} row and the j^{th} column. Further, let

$$A_{i,j} = (-E)^{i+j} A^{i,j} \quad (1 \leq i, j \leq r),$$

where E is the unit element of M . Similarly as in the proof of Cramer's rule, from (7) with some j ($1 \leq j \leq r$)

$$\begin{aligned} A_{1,j} \left(C_{1,1} \left(X^{(1)} \right) + C_{1,2} \left(X^{(2)} \right) + \cdots + C_{1,r} \left(X^{(r)} \right) \right) &= A_{1,j}(0^*) \\ A_{2,j} \left(C_{2,1} \left(X^{(1)} \right) + C_{2,2} \left(X^{(2)} \right) + \cdots + C_{2,r} \left(X^{(r)} \right) \right) &= A_{2,j}(0^*) \\ \vdots \\ A_{r,j} \left(C_{r,1} \left(X^{(1)} \right) + C_{r,2} \left(X^{(2)} \right) + \cdots + C_{r,r} \left(X^{(r)} \right) \right) &= A_{r,j}(0^*) \end{aligned}$$

follows. From this system, using (4) and the fact that the multiplication in the ring of operators is commutative, we get

$$\begin{aligned} C_{1,1}A_{1,j} \left(X^{(1)} \right) + C_{1,2}A_{1,j} \left(X^{(2)} \right) + \cdots + C_{1,r}A_{1,j} \left(X^{(r)} \right) &= 0^* \\ C_{2,1}A_{2,j} \left(X^{(1)} \right) + C_{2,2}A_{2,j} \left(X^{(2)} \right) + \cdots + C_{2,r}A_{2,j} \left(X^{(r)} \right) &= 0^* \\ \vdots \\ C_{r,1}A_{r,j} \left(X^{(1)} \right) + C_{r,2}A_{r,j} \left(X^{(2)} \right) + \cdots + C_{r,r}A_{r,j} \left(X^{(r)} \right) &= 0^* \end{aligned}$$

since $A(0^*) = 0^*$ for any $A \in M$. Adding the equations of this system by (5) we obtain the equation

$$(11) \quad \sum_{t=1}^r (C_{1,t}A_{1,j} + C_{2,t}A_{2,j} + \cdots + C_{r,t}A_{r,j}) \left(X^{(t)} \right) = 0^*.$$

But

$$\sum_{i=1}^r C_{i,t}A_{i,j} = \begin{cases} Z & \text{if } t \neq j \\ \det(C_{i,j}) & \text{if } t = j \end{cases}$$

and so (11) implies (8).

By the Lemma (9) is equivalent to the inequality

$$(12) \quad \det(T(C_{i,j})) \neq 0,$$

where 0 is the identically zero polynomial. But the leading coefficient of the polynomial $\det(T(C_{i,j}))$ is equal to $\det(c_{i,j,m_j})$ and so (12) follows from (3). This completes the proof of the theorem.

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